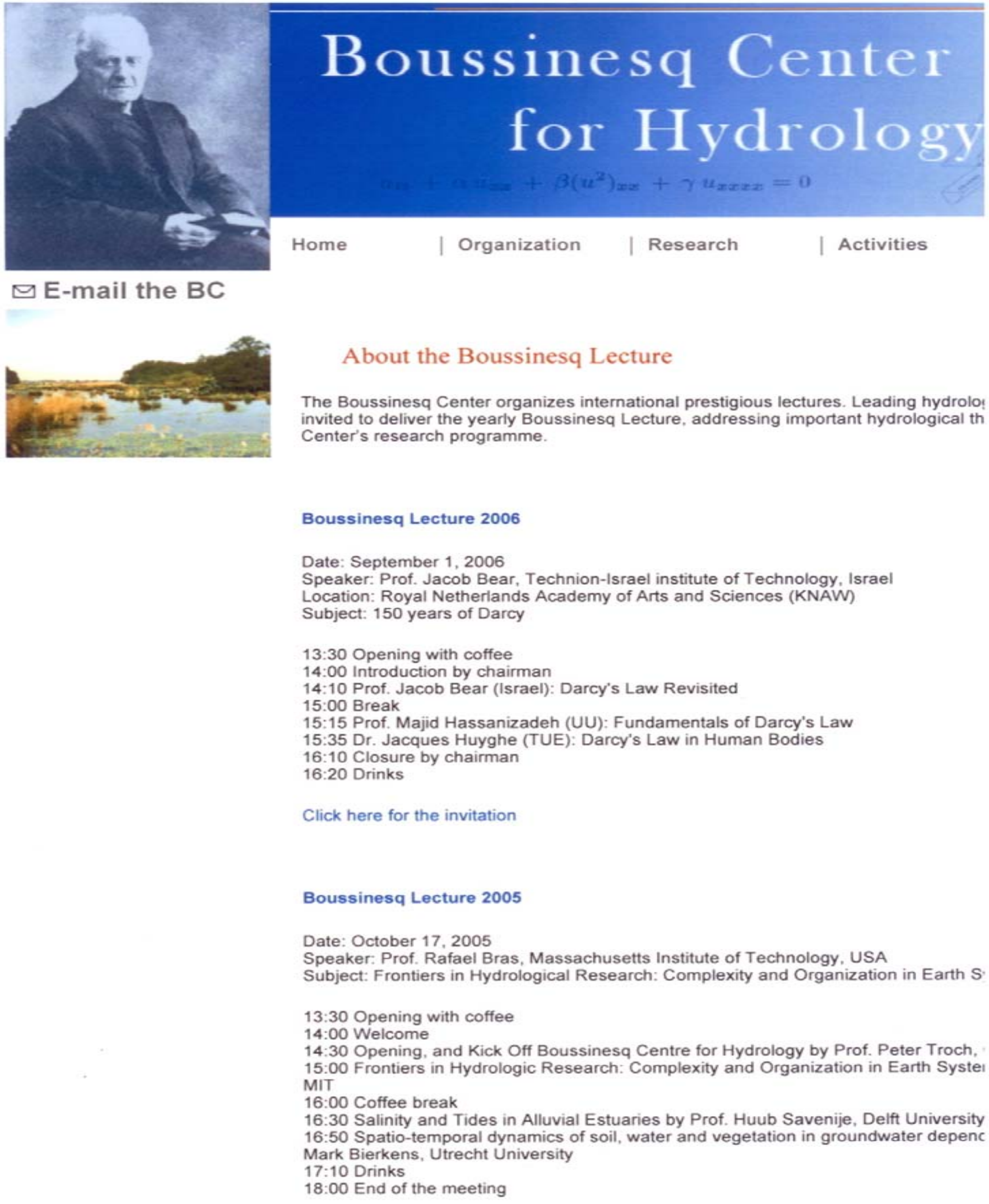


Boussinesq Center for Hydrology

$u_{tt} + \alpha u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0$

About the Boussinesq Lecture

The Boussinesq Center organizes international prestigious lectures. Leading hydrolo
invited to deliver the yearly Boussinesq Lecture, addressing important hydrological th
Center's research programme.

Boussinesq Lecture 2006

Date: September 1, 2006
Speaker: Prof. Jacob Bear, Technion-Israel institute of Technology, Israel
Location: Royal Netherlands Academy of Arts and Sciences (KNAW)
Subject: 150 years of Darcy

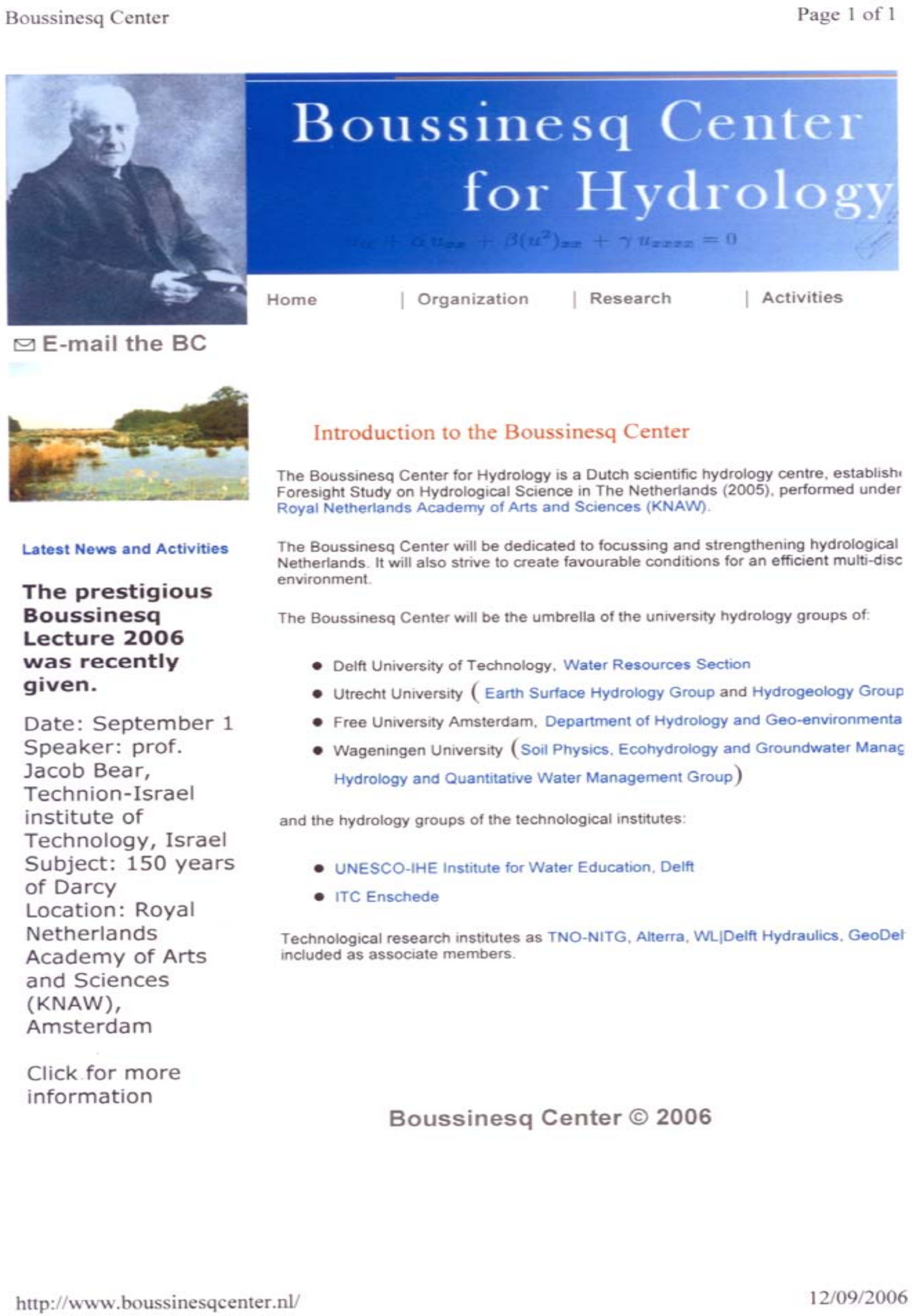
13:30 Opening with coffee
14:00 Introduction by chairman
14:10 Prof. Jacob Bear (Israel): Darcy's Law Revisited
15:00 Break
15:15 Prof. Majid Hassanizadeh (UU): Fundamentals of Darcy's Law
15:35 Dr. Jacques Huyghe (TUE): Darcy's Law in Human Bodies
16:10 Closure by chairman
16:20 Drinks

Click here for the invitation

Boussinesq Lecture 2005

Date: October 17, 2005
Speaker: Prof. Rafael Bras, Massachusetts Institute of Technology, USA
Subject: Frontiers in Hydrological Research: Complexity and Organization in Earth S

13:30 Opening with coffee
14:00 Welcome
14:30 Opening, and Kick Off Boussinesq Centre for Hydrology by Prof. Peter Troch,
15:00 Frontiers in Hydrologic Research: Complexity and Organization in Earth Syste
MIT
16:00 Coffee break
16:30 Salinity and Tides in Alluvial Estuaries by Prof. Huub Savenije, Delft University
16:50 Spatio-temporal dynamics of soil, water and vegetation in groundwater depen
Mark Bierkens, Utrecht University
17:10 Drinks
18:00 End of the meeting



Boussinesq Center Page 1 of 1

Boussinesq Center for Hydrology

$u_{tt} + \alpha u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0$

Home | Organization | Research | Activities

✉ E-mail the BC

Latest News and Activities

The prestigious Boussinesq Lecture 2006 was recently given.

Date: September 1
Speaker: prof. Jacob Bear, Technion-Israel institute of Technology, Israel
Subject: 150 years of Darcy
Location: Royal Netherlands Academy of Arts and Sciences (KNAW), Amsterdam

Click for more information

Introduction to the Boussinesq Center

The Boussinesq Center for Hydrology is a Dutch scientific hydrology centre, establish
Foresight Study on Hydrological Science in The Netherlands (2005), performed under
Royal Netherlands Academy of Arts and Sciences (KNAW).

The Boussinesq Center will be dedicated to focussing and strengthening hydrological
Netherlands. It will also strive to create favourable conditions for an efficient multi-disc
environment.

The Boussinesq Center will be the umbrella of the university hydrology groups of:

- Delft University of Technology, Water Resources Section
- Utrecht University (Earth Surface Hydrology Group and Hydrogeology Group
- Free University Amsterdam, Department of Hydrology and Geo-environmenta
- Wageningen University (Soil Physics, Ecohydrology and Groundwater Manag
  Hydrology and Quantitative Water Management Group)

and the hydrology groups of the technological institutes:

- UNESCO-IHE Institute for Water Education, Delft
- ITC Enschede

Technological research institutes as TNO-NITG, Alterra, WL|Delft Hydraulics, GeoDel
included as associate members.

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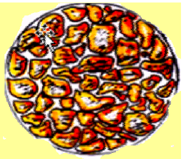
http://www.boussinesqcenter.nl/ 12/09/2006



Boussinesq Center for Hydrology

2nd Boussinesq Lecture

on

DARCY'S LAW---150 years old

1856--2006

by

JACOB BEAR

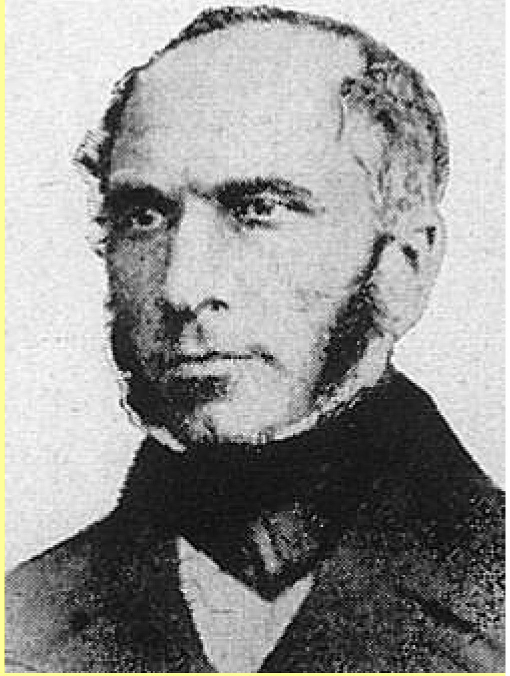
Technion—Israel Institute of Technology

Haifa, Israel

Table of contents

- Who was Henry Darcy?
- Darcy’s Experimental Law
- Extensions
- Upscaling/homogenization/micro to macro
- Darcy’s law as a simplified version of the momentum balance equation—by volume averaging
- Darcy’s law as a simplified version of the momentum balance equation—by homogenization
- Other (non-Darcy) forms of the motion equation
- Single and Multiphase flow models
- What next?

➢Who was Henry Darcy?



- ➢**Henry (Philibert Gaspard) Darcy was born in Dijon, in the Southern part of France, in 1803.**
- ➢**In 1821 he enrolled at the *Ecole Polytechnique* in Paris, and then continued in 1823 to study at the *Ecole des Ponts at Chaussees*.**
- ➢**His studies led to a position with the *Dept. of Bridges & Roads***
- ➢**He met and married (1828) an English woman and established his home in Dijon.**
- ➢**One of his main early projects was the water supply system (pressure pipes) for the city of Dijon, bringing water by a covered aqueduct from the Roster Spring, some 12.7 km from the city, to a reservoir. He was also involved in many other projects, as well as in city politics**

http://en.wikipedia.org/wiki/Image:Henry_Darcy

- **During this period, he modified the Prony equation for calculating the head loss in pipes, due to friction. Later, this equation was further modified by Julius Weisach to become the well known Darcy-Weisbach equation for head losses in pipes.**

- **His lifelong goal was to convert the water supply system of the city of Dijon, which was using highly polluted water from shallow wells and streams, to a centralized water distribution system, which he designed. In 1840 Dijon's public water supply was one of the best in Europe, in terms of both quantity and quality,**

- In 1848 he became the Chief Engineer for the Department of Cote-d`Or (around Dijon). However, due to political pressure, he had to leave to become the Chief Director for Water and Pavements in Paris.

In that position, he could focus on his research on friction losses in pressure pipes, for example, improving the Pitot Tube.

- Due to poor health, he resigned and returned to Dijon in 1855, where he continued his research.

- During 1855-1856, he devoted his research to study the flow of water and the resulting head loss in sand columns. This research led to what we refer to as DARCY's LAW. The motivation for this research: filtration of the water for the fountains of the city of Dijon.

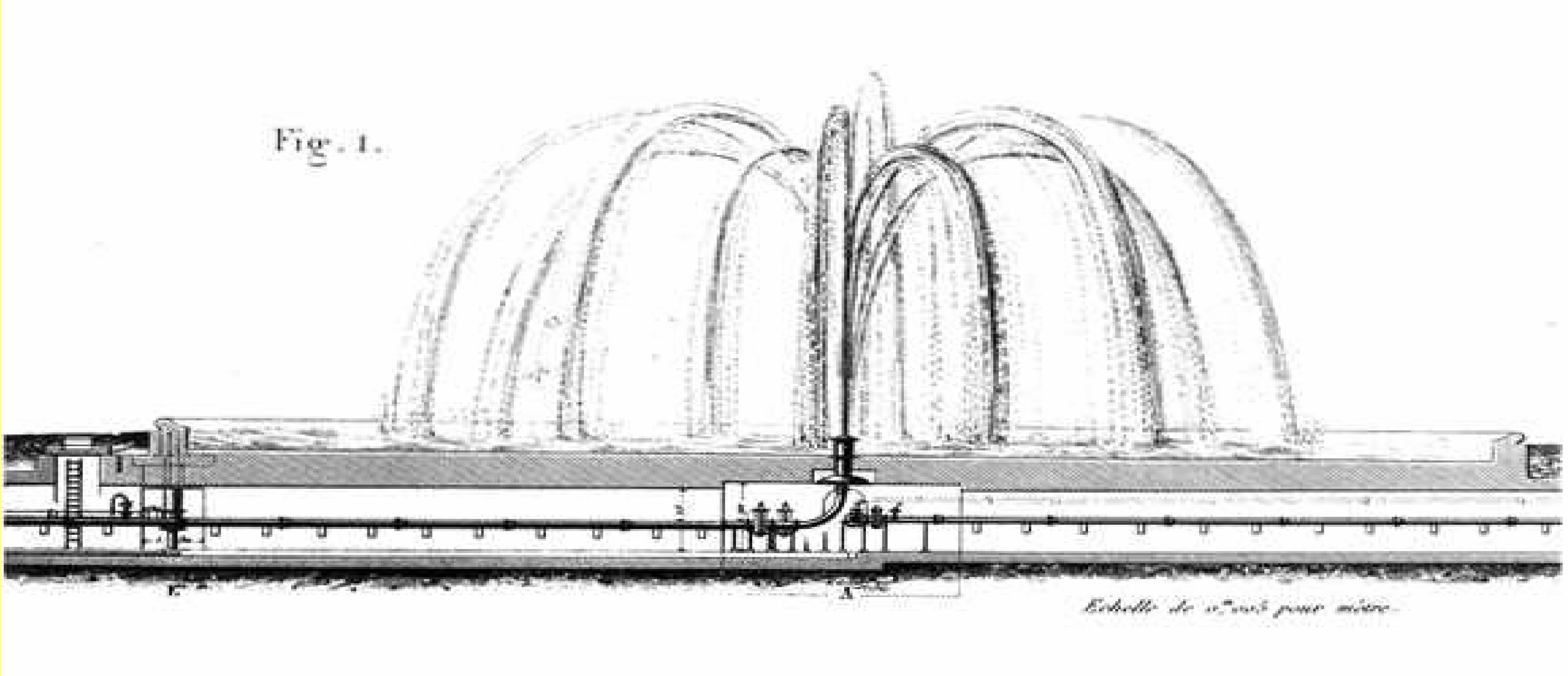
Primarily an engineer, but he loved to conduct research, and did so whenever he felt that the prevailing knowledge was insufficient to solve a problem satisfactorily.

- **His approach to research was purely scientific and in line with the empirical approach to science that was common during at that time. Besides his hydraulic experiments, he developed a theory for recharge by infiltration from rainwater to groundwater, offered a rational explanation for seasonal variation of the productivity of wells, and provided a theoretical explanation for the occurrence of artesian wells, phenomena which were then still issues of serious debate.**

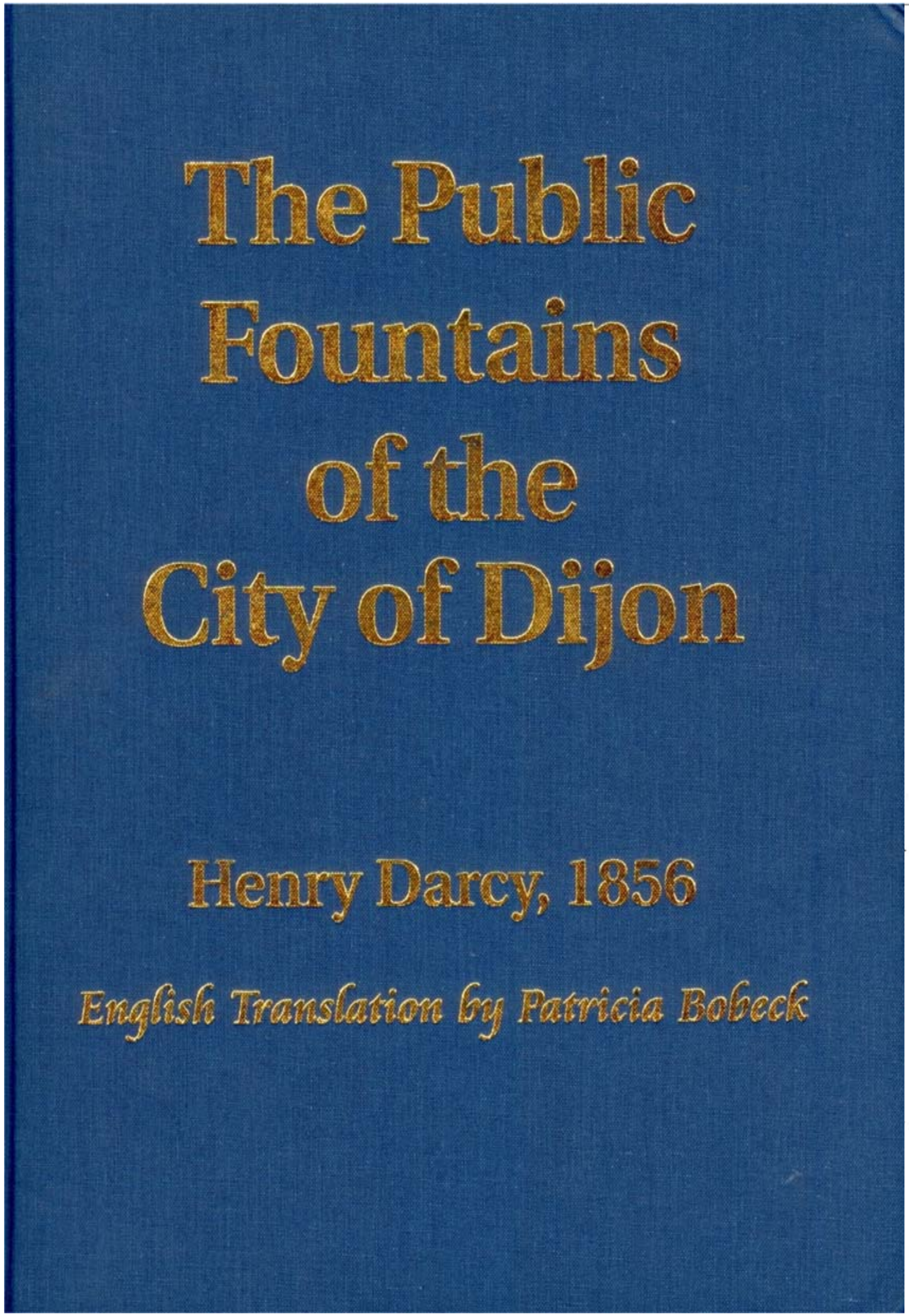
- **Henry Darcy died of pneumonia in 1858 and was buried in Dijon.**

- His research and engineering plans were summarized in his book:

H. Darcy, Les Fontaines Publiques de la Ville de Dijon,
Dalmont, Paris (1856).



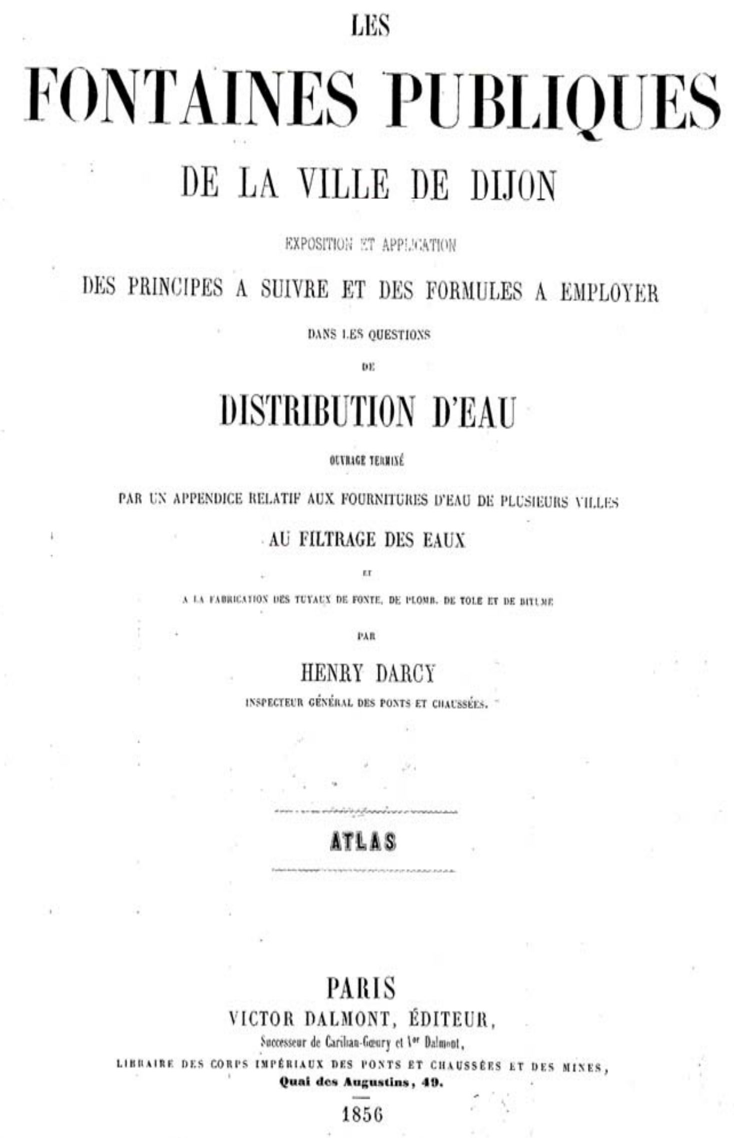
Recently translated into English by Patricia Bobeck



The Public Fountains of the City of Dijon

Henry Darcy, 1856

English Translation by Patricia Bobeck



LES

FONTAINES PUBLIQUES

DE LA VILLE DE DIJON

EXPOSITION ET APPLICATION

DES PRINCIPES A SUIVRE ET DES FORMULES A EMPLOYER

DANS LES QUESTIONS

DE

DISTRIBUTION D'EAU

OUVRAGE TERMINÉ

PAR UN APPENDICE RELATIF AUX FOURNITURES D'EAU DE PLUSIEURS VILLES

AU FILTRAGE DES EAUX

ET

A LA FABRICATION DES TUYAUX DE FONTE, DE PLOMB, DE TOLE ET DE BITUME

PAR

HENRY DARCY

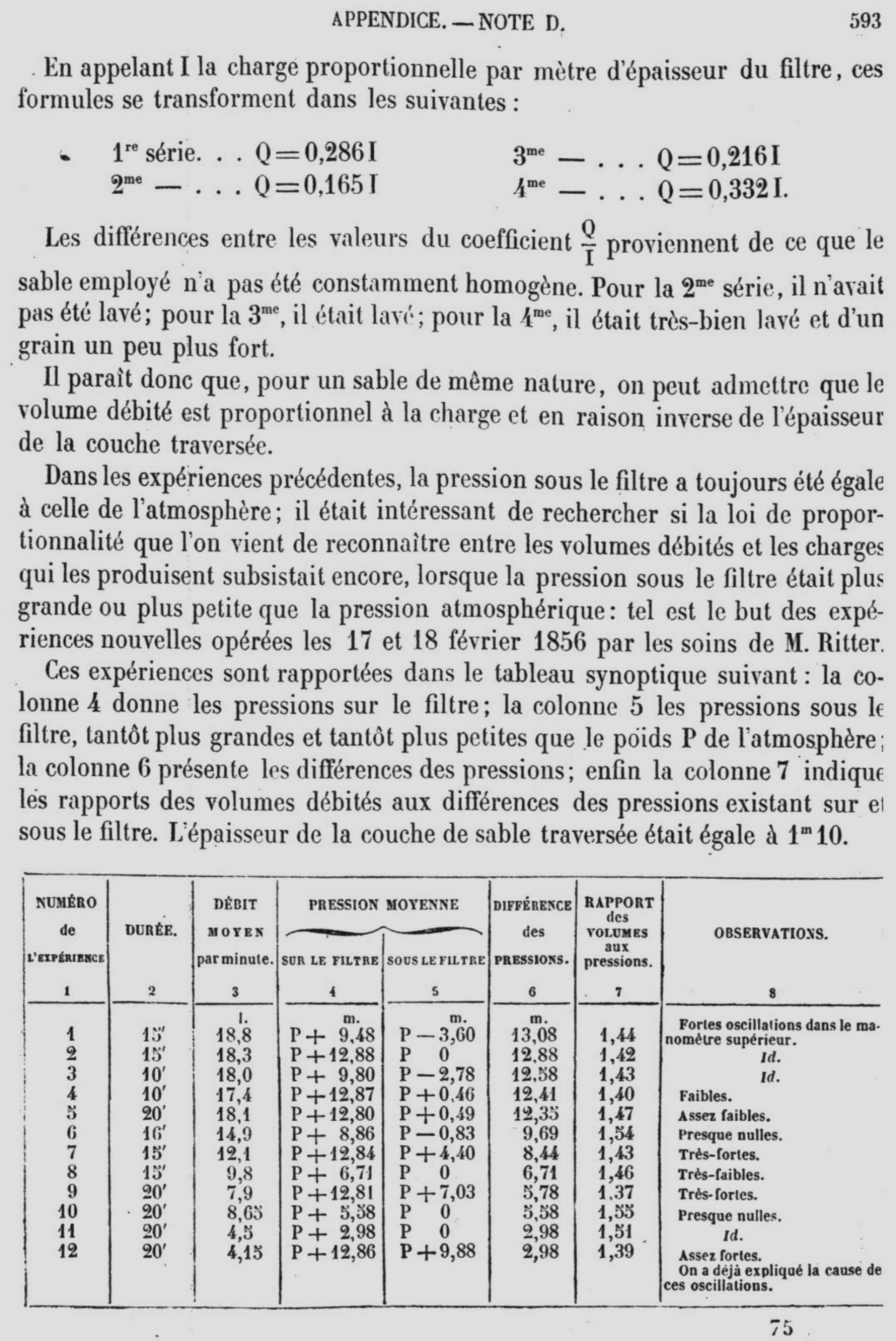
INSPECTEUR GÉNÉRAL DES PONTS ET CHAUSSÉES.

ATLAS

PARIS
VICTOR DALMONT, ÉDITEUR,
Successeur de Carilian-Gœury et V^{ve} Dalmont,
LIBRAIRE DES CORPS IMPÉRIAUX DES PONTS ET CHAUSSÉES ET DES MINES,
Quai des Augustins, 49.

1856

- A page in Darcy’s book in which the experimental data are presented:



APPENDICE. — NOTE D. 593

En appelant I la charge proportionnelle par mètre d'épaisseur du filtre, ces formules se transforment dans les suivantes :

1^{re} série. . . $Q = 0{,}286\,I$ 3^{me} — . . . $Q = 0{,}216\,I$
2^{me} — . . . $Q = 0{,}165\,I$ 4^{me} — . . . $Q = 0{,}332\,I$.

Les différences entre les valeurs du coefficient $\frac{Q}{I}$ proviennent de ce que le sable employé n'a pas été constamment homogène. Pour la 2^{me} série, il n'avait pas été lavé ; pour la 3^{me}, il était lavé ; pour la 4^{me}, il était très-bien lavé et d'un grain un peu plus fort.

Il paraît donc que, pour un sable de même nature, on peut admettre que le volume débité est proportionnel à la charge et en raison inverse de l'épaisseur de la couche traversée.

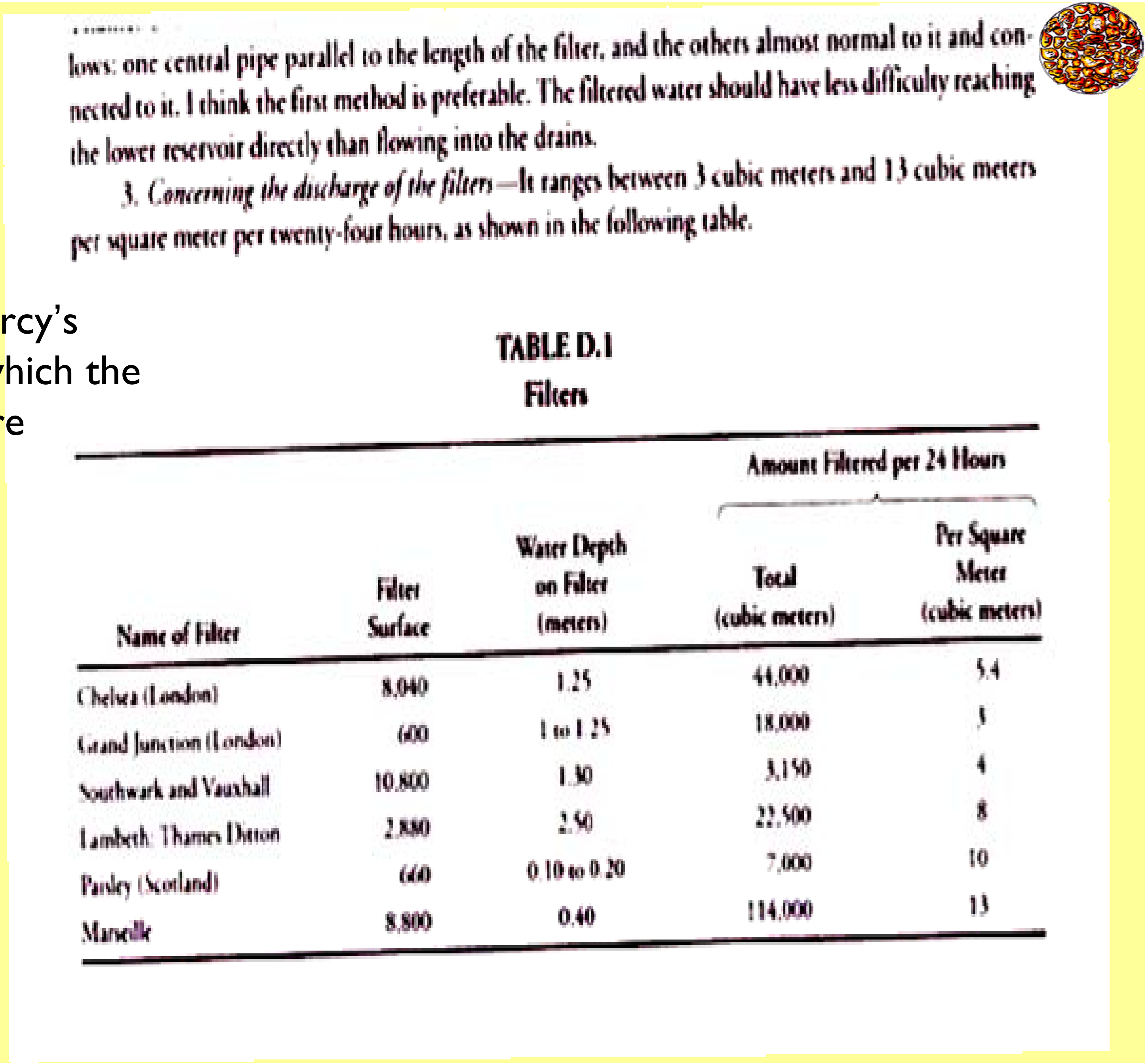
Dans les expériences précédentes, la pression sous le filtre a toujours été égale à celle de l'atmosphère ; il était intéressant de rechercher si la loi de proportionnalité que l'on vient de reconnaître entre les volumes débités et les charges qui les produisent subsistait encore, lorsque la pression sous le filtre était plus grande ou plus petite que la pression atmosphérique : tel est le but des expériences nouvelles opérées les 17 et 18 février 1856 par les soins de M. Ritter.

Ces expériences sont rapportées dans le tableau synoptique suivant : la colonne 4 donne les pressions sur le filtre ; la colonne 5 les pressions sous le filtre, tantôt plus grandes et tantôt plus petites que le poids P de l'atmosphère ; la colonne 6 présente les différences des pressions ; enfin la colonne 7 indique les rapports des volumes débités aux différences des pressions existant sur e sous le filtre. L'épaisseur de la couche de sable traversée était égale à $1^{m}10$.

NUMÉRO de L'EXPÉRIENCE 1	DURÉE. 2	DÉBIT MOYEN par minute. 3	PRESSION MOYENNE SUR LE FILTRE 4	PRESSION MOYENNE SOUS LE FILTRE 5	DIFFÉRENCE des PRESSIONS. 6	RAPPORT des VOLUMES aux pressions. 7	OBSERVATIONS. 8
		l.	m.	m.	m.		
1	15′	18,8	P + 9,48	P — 3,60	13,08	1,44	Fortes oscillations dans le manomètre supérieur.
2	15′	18,3	P + 12,88	P 0	12,88	1,42	*Id.*
3	10′	18,0	P + 9,80	P — 2,78	12,58	1,43	*Id.*
4	10′	17,4	P + 12,87	P + 0,46	12,41	1,40	Faibles.
5	20′	18,1	P + 12,80	P + 0,49	12,35	1,47	Assez faibles.
6	16′	14,9	P + 8,86	P — 0,83	9,69	1,54	Presque nulles.
7	15′	12,1	P + 12,84	P + 4,40	8,44	1,43	Très-fortes.
8	15′	9,8	P + 6,71	P 0	6,71	1,46	Très-faibles.
9	20′	7,9	P + 12,81	P + 7,03	5,78	1,37	Très-fortes.
10	20′	8,65	P + 5,58	P 0	5,58	1,55	Presque nulles.
11	20′	4,5	P + 2,98	P 0	2,98	1,51	*Id.*
12	20′	4,15	P + 12,86	P + 9,88	2,98	1,39	Assez fortes. On a déjà expliqué la cause de ces oscillations.

75

- Another page of Darcy’s translated book in which the experimental data are presented:



lows: one central pipe parallel to the length of the filter, and the others almost normal to it and connected to it. I think the first method is preferable. The filtered water should have less difficulty reaching the lower reservoir directly than flowing into the drains.

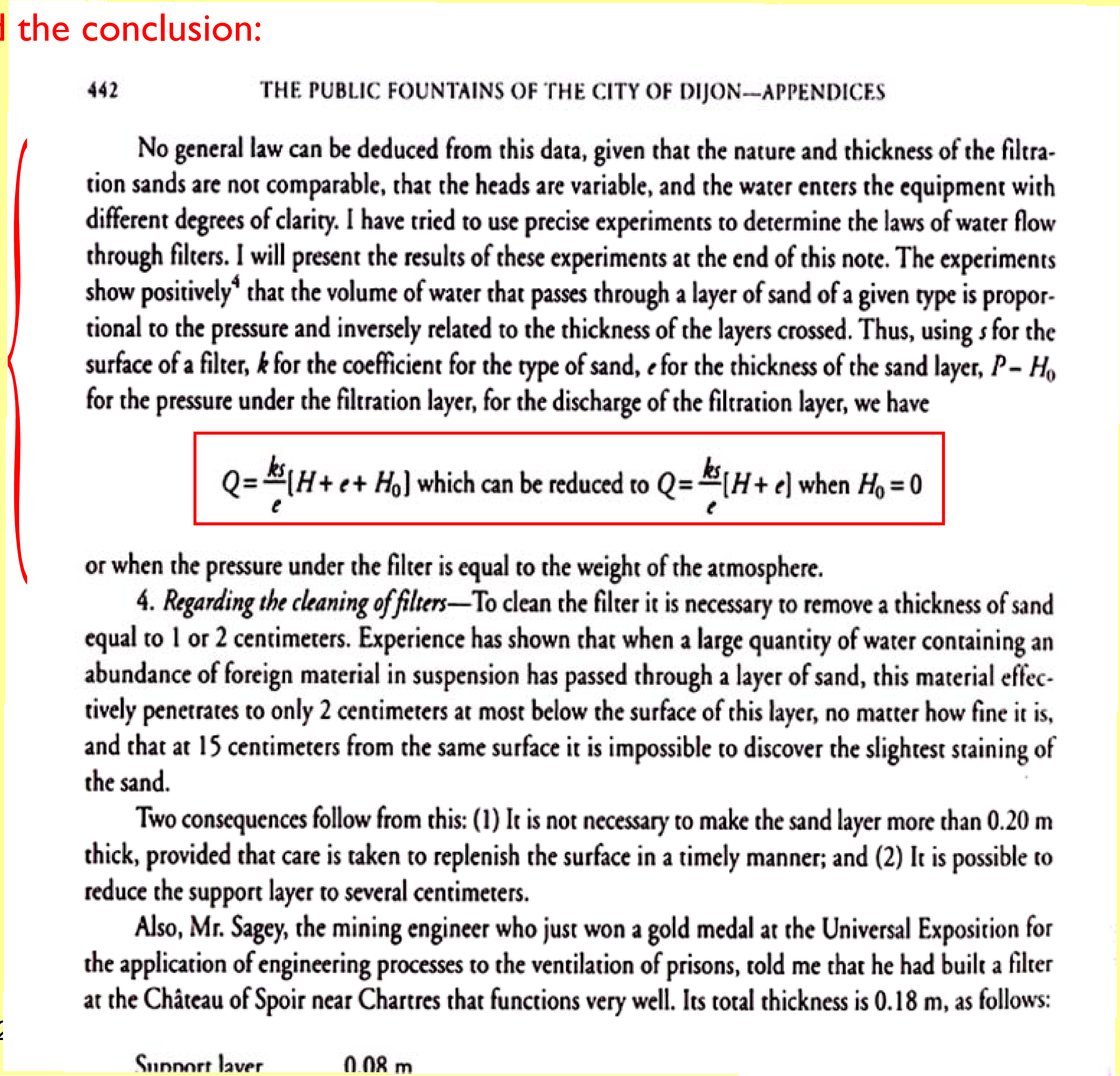
3. *Concerning the discharge of the filters*—It ranges between 3 cubic meters and 13 cubic meters per square meter per twenty-four hours, as shown in the following table.

TABLE D.1

Filters

Name of Filter	Filter Surface	Water Depth on Filter (meters)	Amount Filtered per 24 Hours Total (cubic meters)	Amount Filtered per 24 Hours Per Square Meter (cubic meters)
Chelsea (London)	8,040	1.25	44,000	5.4
Grand Junction (London)	600	1 to 1.25	18,000	3
Southwark and Vauxhall	10,800	1.30	3,150	4
Lambeth: Thames Ditton	2,880	2.50	22,500	8
Paisley (Scotland)	660	0.10 to 0.20	7,000	10
Marseille	8,800	0.40	114,000	13

…and the conclusion:



No general law can be deduced from this data, given that the nature and thickness of the filtration sands are not comparable, that the heads are variable, and the water enters the equipment with different degrees of clarity. I have tried to use precise experiments to determine the laws of water flow through filters. I will present the results of these experiments at the end of this note. The experiments show positively[4] that the volume of water that passes through a layer of sand of a given type is proportional to the pressure and inversely related to the thickness of the layers crossed. Thus, using s for the surface of a filter, k for the coefficient for the type of sand, e for the thickness of the sand layer, $P - H_0$ for the pressure under the filtration layer, for the discharge of the filtration layer, we have

$$\boxed{Q=\frac{ks}{e}[H+e+H_0] \text{ which can be reduced to } Q=\frac{ks}{e}[H+e] \text{ when } H_0=0}$$

or when the pressure under the filter is equal to the weight of the atmosphere.

4. *Regarding the cleaning of filters*—To clean the filter it is necessary to remove a thickness of sand equal to 1 or 2 centimeters. Experience has shown that when a large quantity of water containing an abundance of foreign material in suspension has passed through a layer of sand, this material effectively penetrates to only 2 centimeters at most below the surface of this layer, no matter how fine it is, and that at 15 centimeters from the same surface it is impossible to discover the slightest staining of the sand.

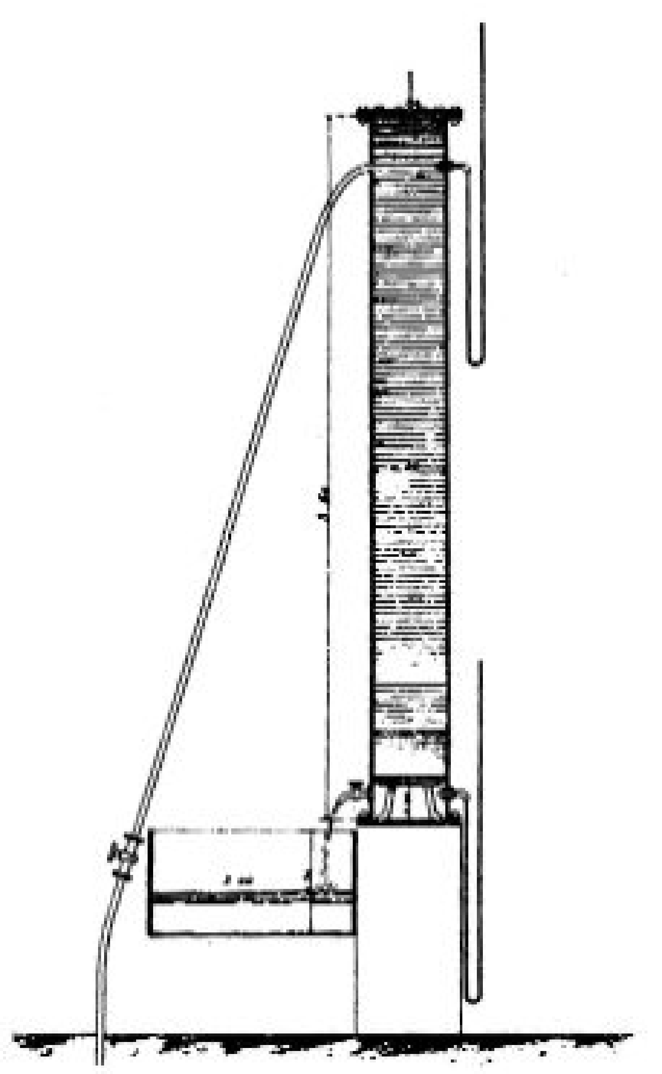
Two consequences follow from this: (1) It is not necessary to make the sand layer more than 0.20 m thick, provided that care is taken to replenish the surface in a timely manner; and (2) It is possible to reduce the support layer to several centimeters.

Also, Mr. Sagey, the mining engineer who just won a gold medal at the Universal Exposition for the application of engineering processes to the ventilation of prisons, told me that he had built a filter at the Château of Spoir near Chartres that functions very well. Its total thickness is 0.18 m, as follows:

Support layer 0.08 m

DARCY's (EXPERIMENTAL) LAW

- The sand column experimental setup.



Flow rate, *Q*, is proportional to:
head difference, $\Delta h = h_1 - h_2$
cross-sectional area, A,
and inversely proportional to
length of column, L.

↓

$$Q = KA\frac{h_1 - h_2}{L},$$

in which K is a coefficient of proportionality that depends on the permeability of the sand.

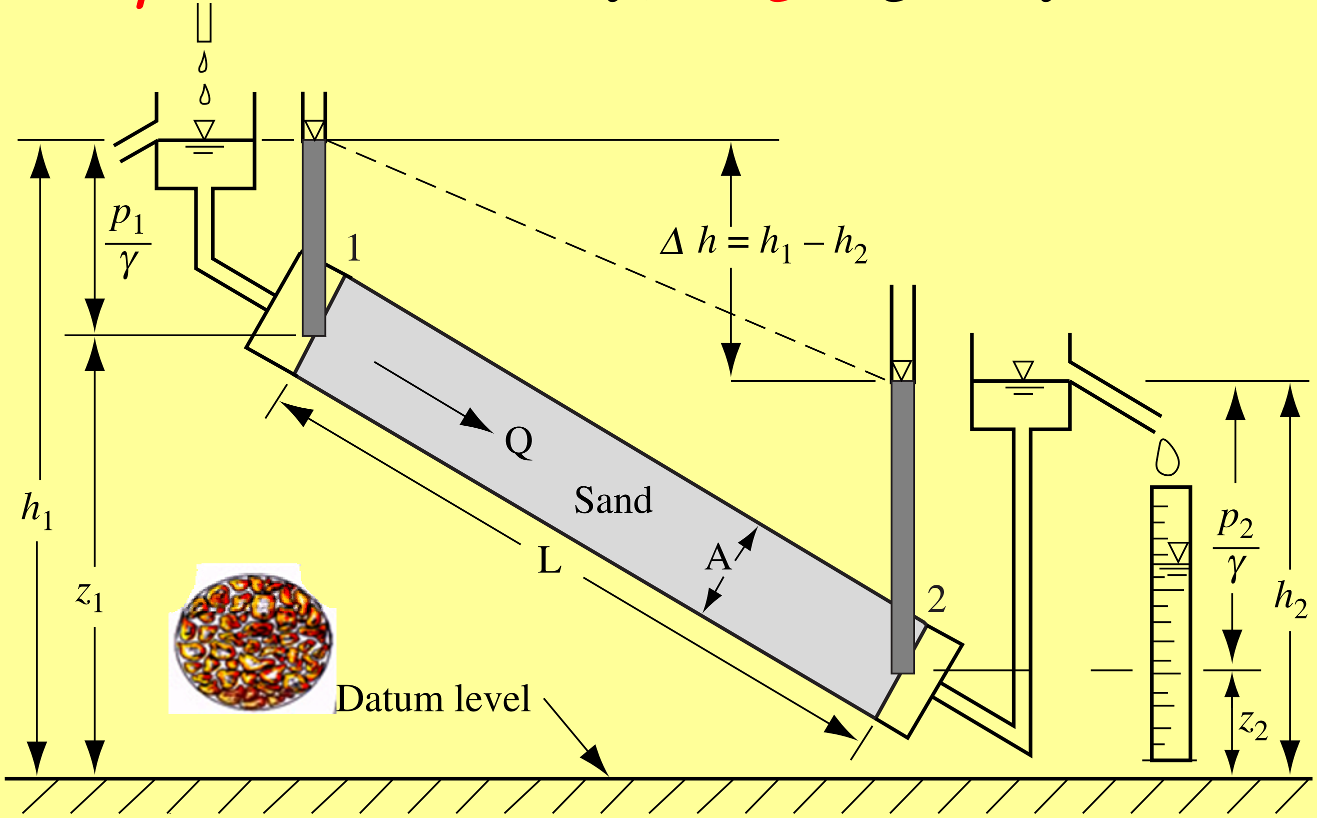
- (Now) We refer to K as the hydraulic conductivity of the porous medium.

Not original symbols

$$Q = KA\frac{h_1 - h_2}{L}$$

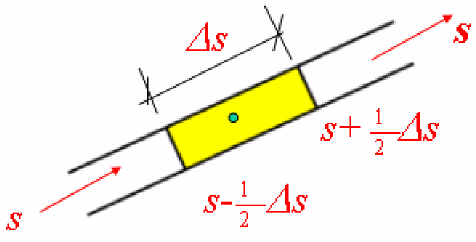
$h =$ piezometric head $\qquad h = z + \dfrac{p}{\rho g}$

$\rho =$ mass density; $\quad g$ = gravity accelaration.



- At a point: HYDRAULIC GRADIENT:

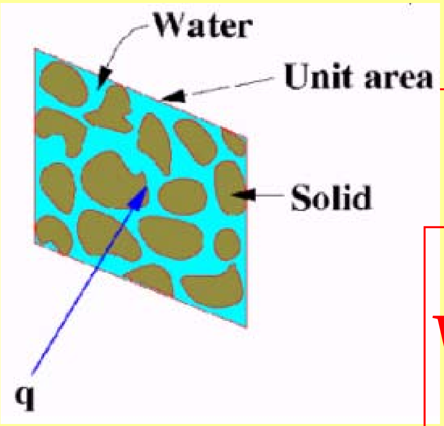
$$\mathcal{J} = \frac{h_1 - h_2}{L} \equiv \frac{h(s) - h(s+\Delta s)}{L} = -\frac{dh(s)}{ds}$$

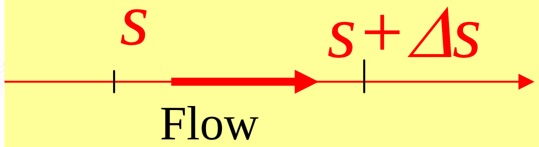


$$\boxed{q = \frac{Q}{A} = \text{specific discharge}}$$

$= \text{volume of water per unit time}$

through a unit area normal to the flow.





$$\boxed{V = \frac{q}{\phi} = \begin{cases} \text{water} \\ \text{velocity} \end{cases}}$$

….OVER THE YEARS….EXTENSIONS…

- Compressible fluid $\rho = \rho(p)$

$$h^* = h^*(\mathbf{x}, t) = z + \int_{p_o}^{p} \frac{dp}{g\rho(p)}$$

(Hubbert's potential)

- Permeability, k, vs. Hydraulic conductivity, K.

$$K = k\frac{\rho g}{\mu},$$

μ = dynamic viscosity

- For one-dimensional flow: $q_s = -K \dfrac{dh(s)}{ds}$

- For three- dimensional flow; isotropic porous medium

$$\mathbf{q} = -K\nabla h \equiv K\mathcal{J}, \quad \mathbf{q}, \mathcal{J} = \text{vectors}, \; \mathcal{J}$$ = hydraulic gradient

- Or:

$$q_i = K\mathcal{J}_i \quad i = 1,2,3 \quad (or\ x, y, z), \quad \mathcal{J}_i = -\frac{dh}{dx_i}$$

$$\mathcal{J} \equiv -\nabla h \text{..........Hydraulic gradient}$$

- For three- dimensional flow; anisotropic porous medium

$$q_x = K_{xx}\mathcal{J}_x + K_{xy}\mathcal{J}_y + K_{xz}\mathcal{J}_z$$

$$q_y = K_{yx}\mathcal{J}_x + K_{yy}\mathcal{J}_y + K_{yz}\mathcal{J}_z$$

$$q_z = K_{zx}\mathcal{J}_x + K_{zy}\mathcal{J}_y + K_{zz}\mathcal{J}_z$$

$$\mathrm{K} = \begin{vmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{vmatrix}$$

..second rank tensor

...principal directions, etc.

$$\mathbf{q} = \mathbf{K}\cdot\mathcal{J}, \quad q_i = K_{ij}\frac{dh}{dx_j} \quad \text{(summation convention applicabe)}$$

- Inhomogeneous porous medium:

We use $K = K(x,y,z)$ in Darcy’s Law.

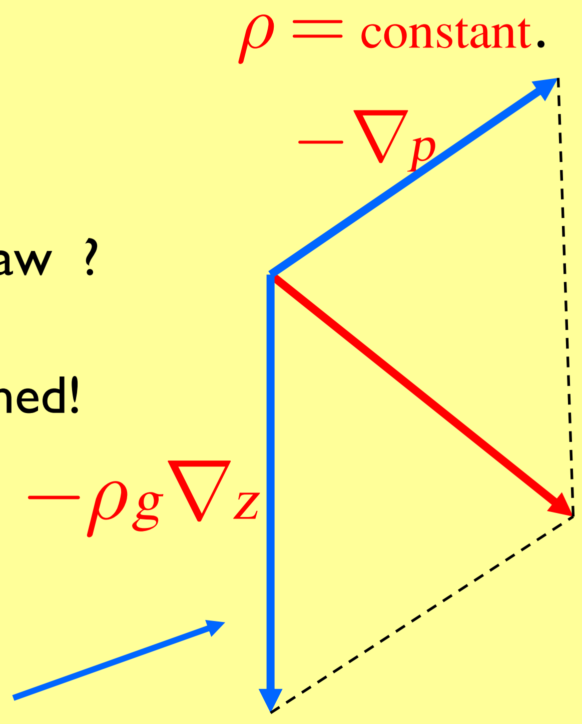
- Inhomogeneous fluid: $\rho = \rho(p, c, T)$

Direct extension from the original Darcy Law ?

The piezometric head , h ,cannot be defined!

- We extend to $\rho \neq \text{constant.}$

$$\mathbf{q} = -\frac{k}{\mu}(\nabla p + \rho g \nabla z)$$

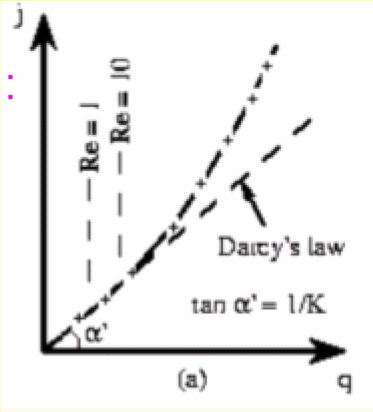


Driving forces

- We’ll see later that this is indeed the generalized form of Darcy’s Law

Range of validity of Darcy’s (linear) law:

- Experiments.



- By analogy to flow through pipes:

Reynolds number: $\mathrm{Re} = \frac{qd}{\nu}, \quad \nu = \frac{\mu}{\rho g} = \text{kinematic viscosity.}$

d = some representative (microscopic) length characterizing void space, e.g. d_{10}.

- Darcy's law is valid as long as the Re, that indicates the magnitude of the inertial forces relative to the viscous drag ones, does not exceed a value of about 1 (but sometimes as high as 10).

$$\mathrm{Re} \ll 1.$$

- So far, (the experimentally derived) Darcy's law looks like a constitutive relation, Or a phenomenological law, with a coefficient (*k*) that represents material propert and has to be obtained experimentally.

- There have been many attempts to "derive" Darcy's law, either by analogy to laminar flow in a network of pipes, or by studying the resistance to laminar flow around an individual grain. From such investigations, conclusions were drawn with respect to a porous medium, visualized as a network of tortuous capillary pipes .

Using Hagen-Poisseuille's law:

$$Q_s = -\frac{\pi d^4}{128}\frac{\rho g}{\mu}\frac{dh}{ds} \Rightarrow V_s = -\frac{d^2}{32}\frac{\rho g}{\mu}\frac{dh}{ds}$$

$$q_s = \frac{Q_s}{ab} = -N\frac{\pi d^4}{128}\frac{\rho g}{\mu}\frac{dh}{ds} \Rightarrow q_s = -k\frac{\rho g}{\mu}\frac{dh}{ds}$$

$$k = -\phi\frac{d^2}{32}$$

Etc.

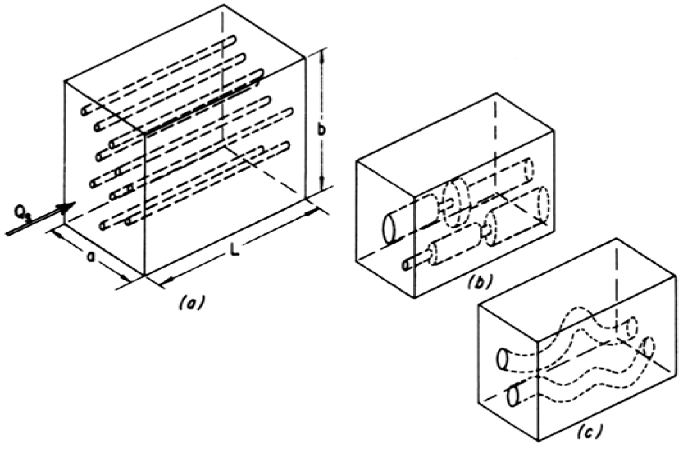
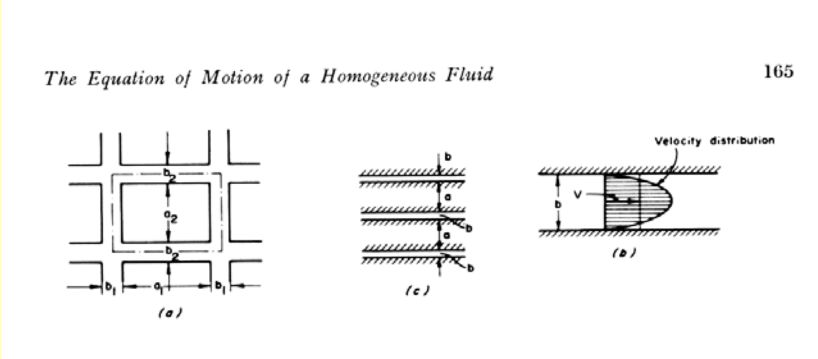


FIG. 5.10.1. Capillary tube models.

- Fissure models:

$$V = \frac{b^2}{12}\frac{\rho g}{\mu}\frac{dh}{ds} \Rightarrow q = k\frac{\rho g}{\mu}\frac{dh}{ds}$$

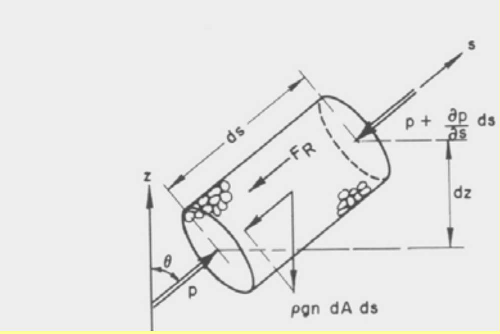
$$k = \frac{\phi^3}{(1-\phi)^2}\frac{a^2}{12}$$



- Resistance to flow models

Drag on an individual particle in laminar flow (Stokes):

$$D = 3\pi d\mu V = \frac{24}{\mathrm{Re}}\rho\frac{V^2}{2}A = C_D\rho\frac{V^2}{2}\frac{\pi d^2}{4}$$



….balance of forces…drag, pressure, gravity..

$$q_s = -\frac{k}{\mu}\left(\frac{\partial p}{\partial s} + \rho g\frac{\partial z}{\partial s}\right)$$

- ALL BASED ON…..the NAVIER-STOKES equation (for a Newtonian fluid)..

….conclusion…??

- There have been attempts to extend Darcy's law for higher Reynolds number, ("correction term"), e.g.,

- Forchheimer (1901): $J = Wq + bq^m, \quad 1.6 \le m \le 2$

- Kozeny-Carman (Scheidegger, 1960): $J = 180\alpha \frac{(1-\phi)^2 \nu}{g\phi^3 d^2} q + \frac{3\beta(1-\phi)}{4g\phi^3 d} q^2$

- Ward (1964): $J = \frac{\nu}{gk} q + \frac{0.55}{g\sqrt{k}} q^2, \quad k = \frac{d^2}{360}$

- Joseph *et al (1982)* $\nabla p = \frac{\mu}{k}\mathbf{q} + c_F \frac{1}{\sqrt{k}} \rho |\mathbf{q}| \mathbf{q}$

Hagen – Poisseuille′s law

Stokes law

Hele – Shaw model (parallel plates)

⇒ All are simplified forms of the Navier stokes equation (Newtonian fkuid), or the momentum balance equation.

Darcy's law in not a constitutive law….

- We turn our attention to the momentum balance equation.

But first:

What is wrong in this table?

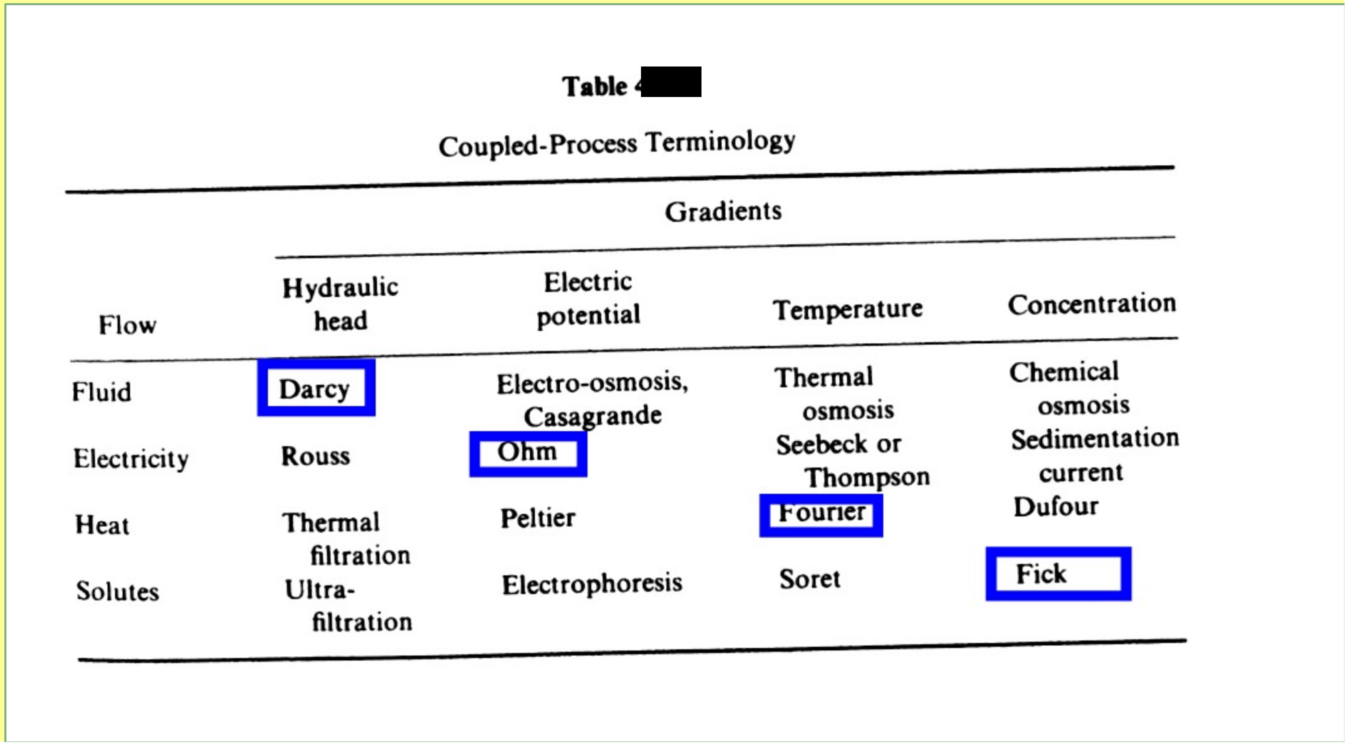
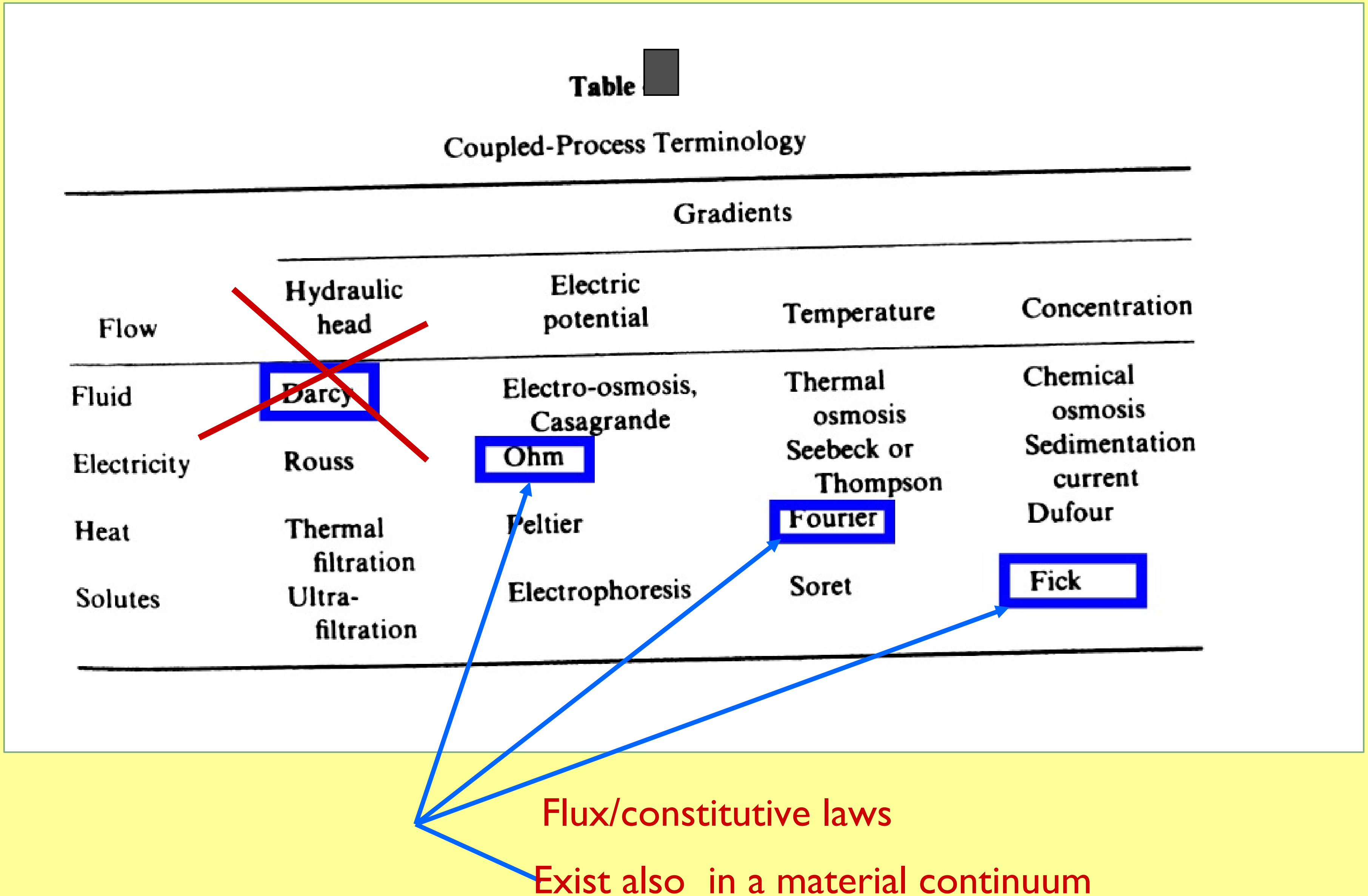


Table [redacted]

Coupled-Process Terminology

Flow	Gradients			
	Hydraulic head	Electric potential	Temperature	Concentration
Fluid	Darcy	Electro-osmosis, Casagrande	Thermal osmosis	Chemical osmosis
Electricity	Rouss	Ohm	Seebeck or Thompson	Sedimentation current
Heat	Thermal filtration	Peltier	Fourier	Dufour
Solutes	Ultra-filtration	Electrophoresis	Soret	Fick

What is wrong in this table?



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Heat	Thermal filtration	Peltier	Fourier	Dufour
Solutes	Ultra-filtration	Electrophoresis	Soret	Fick

Flux/constitutive laws

Exist also in a material continuum (microscopic level)

- Recall: This presentation is not on

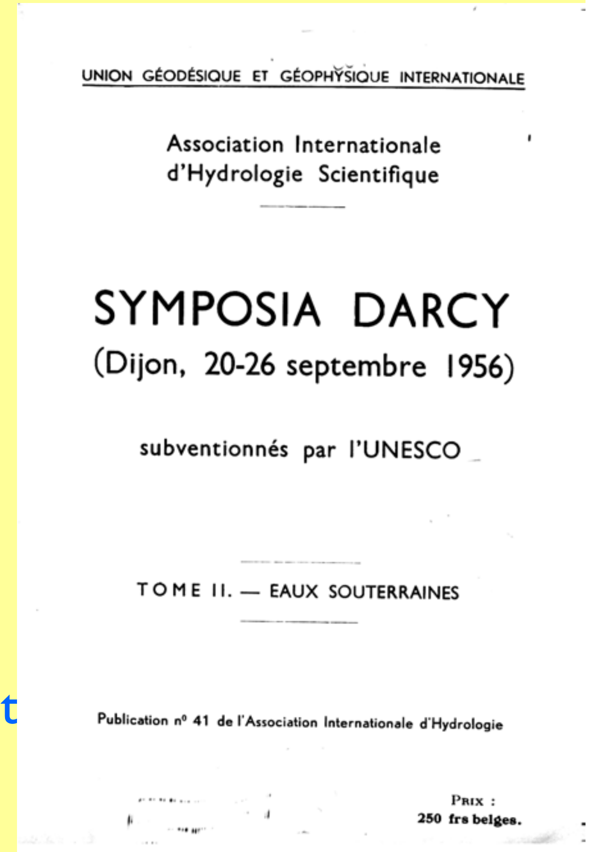
PHENOMENA OF TRANSPORT IN POROUS MEDIA,

But, primarily… around DARCY"S law (and its contribution to…modeling)

….1856……1956: IAHR SYMPOSIA DARCY, Dijon, 20-26, September, 1956.

What was discussed 50 years ago under the subject of "GROUNDWATER" ?

- SCHNEEBELI, G.,: Hydraulics of wells, using the Dupuit assumption (1863), the existence of a seepage, radius of influence, anisotropy…
- KOLLIS, W. (Poland): Flow affected by a cutoff wall- experiments in a sand box (DARCY's law: v=ki)
- Goldschmidt, M.J. and Jacobs M. (Israel). Ground water in the Haifa-Acco dune area.
- IRMAY, S. (Israel) Extension of Dracy's Law to unsteady and unsaturated flow. (citing earlier works by Muskat Richards, Gardner, and the work Edelfson and Anderson who introduced the capillary and osmotic potentials).



UNION GÉODÉSIQUE ET GÉOPHYSIQUE INTERNATIONALE

Association Internationale
d'Hydrologie Scientifique

SYMPOSIA DARCY
(Dijon, 20-26 septembre 1956)

subventionnés par l'UNESCO

TOME II. — EAUX SOUTERRAINES

Publication n° 41 de l'Association Internationale d'Hydrologie

PRIX :
250 frs belges.

- SCHOLLER, H. On the radius of wells and DARCY's COEFFICIENT in a sandy aquifer.

- TANAKA, H. and YAHAGI, F. Flow under a cutoff wall—quantitative considerations

- SCHIFF, I. Darcy's law in planning artificial recharge through a pond.

- MOSONI, E., and KOVACS, G. Similarity laws for sand-box modeling.

- MENETH, E., On the importance of Darcy's law in sand-box modeling.

- UBELL, K., Unsteady flow to wells.

DISPERSION!!

- De JOSSELIN de JONG, G. On the movement of tracer particles in flow through the porous media…..the beginning of his work on dispersion. ….using 3-d random walk….breakthrough curves in a sand column to determine the coefficient of dispersion. A pioneering work!!

Actually, an extension of Darcy's law

also:….. Evaluation of permeability….

…. Hydraulics of wells……

Two phase flow and relative permeabilities to oil and water…

- DAVID, K. TODD, On laboratory research with ground water models..

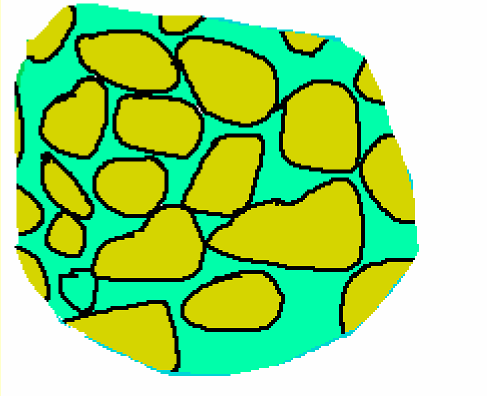
- Tison, G. Jr. On water level fluctuations..

- Stallman, R.W. Numerical methods for analyzing data on ground water levels.

…and additional presentations on such subjects as:

- On flow to horizontal wells
- On pumping tests
- Flow in the vicinity of wells
- Evaluation of ground water resources

…of course, not based on a comprehensive literature survey, but….
but very few discussions (in that symposium) on the following subjects:

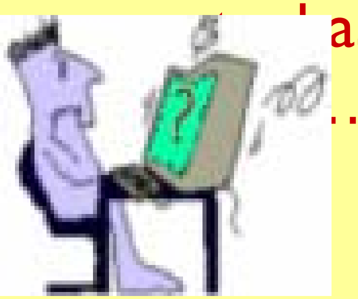


➢Porous medium as a continuum …averaging,

…and …

➢Modelling (conceptual,
mathemaitical,
numerical,
computer solutions…)

➢Effect of heterogeneity…and uncertainty
…. astic modeling
….contamination….etc



THE CONTINUUM APPROACH TO THE DESCRIPTION/MODELING Of PHENOMENA OF TRANSPORT IN POROUS MEDIA.

How is this relvant ?

- What is a continuum?

  A spatial domain at every point of which we can define state variables and material coefficients.

- What is a porous medium?

  A spatial domain containing a solid matrix and a void space (occupied by one or more fluid phases), for which an REV can be determined.

- What is an REV? A spatial domain (sample) such that no matter where you place it within a porous medium domain, it will always contain both a persistent solid phase and a void space.
- Why do we need the CONTINUUM APPROAH to describe/model phenomena of transport in porous media?
- We can describe/model phenomena at the microscopic level (= behavior within a solid or fluid phase), but, practically, we cannot describe the interface between them, To overcome this obstacle, we transform the domain and the phenomena that occur within it to the macroscopic level, at which these inter-phase boundaries do not exist----they are replaced by coefficients that represent at the macroscopic level the effects of these (microscopic level) boundaries

(Molecular) → Microscopic → Macroscopic → Megascopic….

μREV $\qquad$ REV

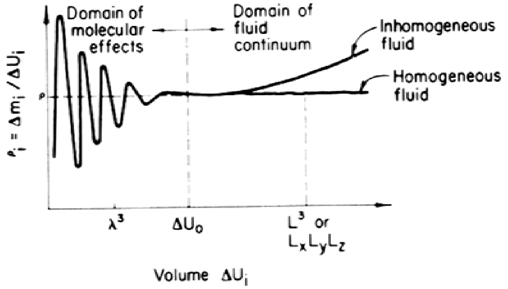
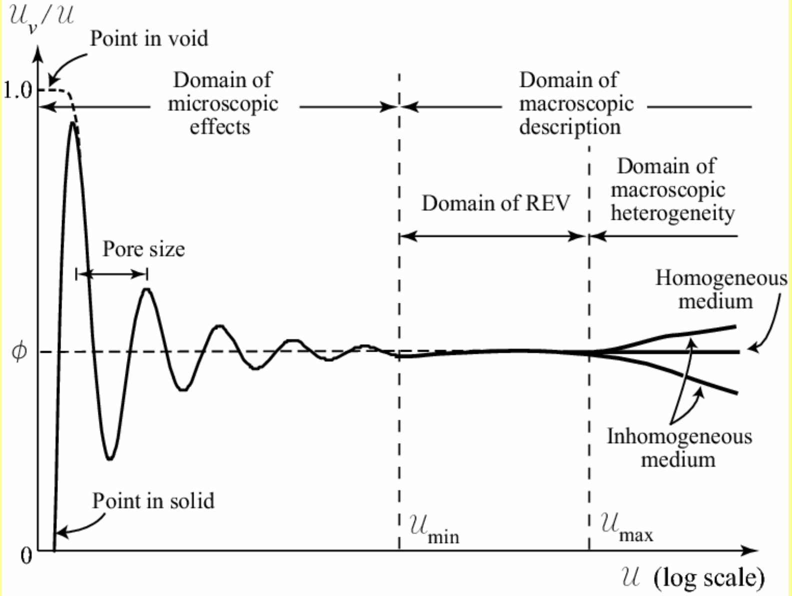
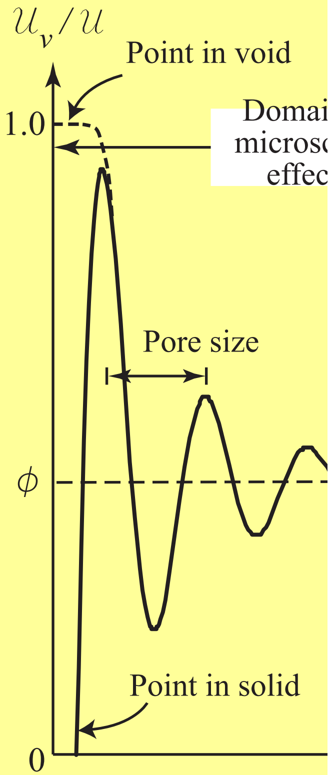


FIG. 1.3.1. Definition of fluid density.



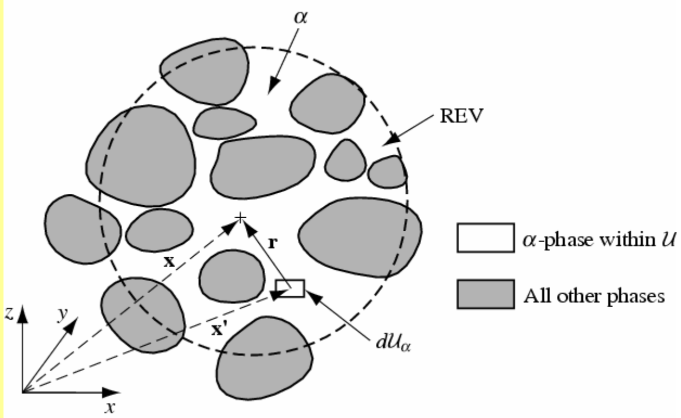


- ➢Averaging over an REV
- ➢Intrinsic phase average

$$\overline{e_\alpha}^{\alpha}(\mathbf{x},t)=\frac{1}{\mathcal{U}_{o\alpha}(\mathbf{x},t)}\int_{\mathcal{U}_{o\alpha}(\mathbf{x},t)} e_\alpha(\mathbf{x}',t;\mathbf{x})\,d\mathcal{U}_\alpha(\mathbf{x}'),$$

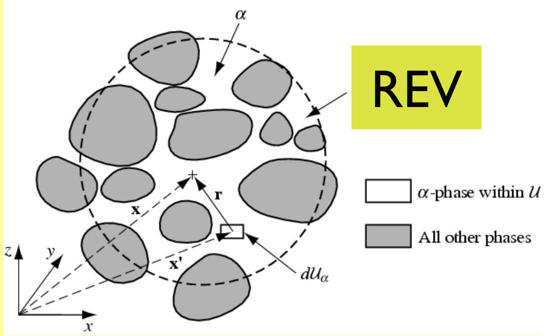
or

$$\overline{e_\alpha}^{\alpha}(\mathbf{x},t)=\frac{1}{\mathcal{U}_{o\alpha}(\mathbf{x},t)}\int_{\mathcal{U}_{o}(\mathbf{x},t)} e_\alpha(\mathbf{x}',t;\mathbf{x})\,\gamma(\mathbf{x}')d\mathcal{U}(\mathbf{x}')$$



Intrinsic phase average

$$\overline{e}_{\alpha}^{\alpha}(\mathbf{x},t)=\frac{1}{\mathcal{U}_{o\alpha}(\mathbf{x},t)}\int_{\mathcal{U}_{o\alpha}(\mathbf{x},t)} e_{\alpha}(\mathbf{x}',t;\mathbf{x})\,d\mathcal{U}_{\alpha}(\mathbf{x}'),\quad \text{or}$$



REV

Phase average

$$\overline{e}_{\alpha}^{\alpha}(\mathbf{x},t)=\frac{1}{\mathcal{U}_{o\alpha}(\mathbf{x},t)}\int_{\mathcal{U}_{o}(\mathbf{x},t)} e_{\alpha}(\mathbf{x}',t;\mathbf{x})\,\gamma(\mathbf{x}')\,d\mathcal{U}(\mathbf{x}'),\quad \text{or}$$

SIZE of REV?

Mass average

$$\left\langle e_{\alpha}{}' \right\rangle=\frac{1}{m_{\alpha}}\int_{m_{\alpha}(\mathbf{x},t)} e_{\alpha}{}'(\mathbf{x}',t)\,dm_{\alpha}(\mathbf{x}')$$

$$=\frac{1}{\rho_{\alpha}\mathcal{U}_{o\alpha}}\int_{\mathcal{U}_{o\alpha}(\mathbf{x},t)} \rho_{\alpha}(\mathbf{x}',t)e_{\alpha}{}'(\mathbf{x}',t)\,d\mathcal{U}_{\alpha}(\mathbf{x}')=\frac{\overline{e}_{\alpha}^{\alpha}}{\overline{\rho}_{\alpha}^{\alpha}}$$

Deviation from the average

$$\overset{o}{e}_{\alpha}(\mathbf{x}',t)\equiv e_{\alpha}(\mathbf{x}',t)-\overline{e}_{\alpha}^{\alpha};\qquad \overline{\overset{o}{e}_{\alpha}}^{\alpha}(\mathbf{x}',t)=0.$$

Average of deviation vanishes

- OBJECTIVE: By averaging (smoothing) over an REV, we overcome the difficulties inherent in the fact that the domain is heterogeneous at the MICROSCOPIC SCALE (presence of solids and pores), **and** we lack information on the geometrical details of this heterogeneity.
- INSTEAD,

  we pass to a level of description (and measurement !), at which this information is not required…..The MACROSCOPIC SCALE (which may still be heterogeneous (e.g., with respect to ***K***).

- The price for this convenience….the creation of (geometrical) coefficients, which reflect at the macroscopic level, the effects of the microscopic inter-phase boundaries that were eliminated by the averaging:

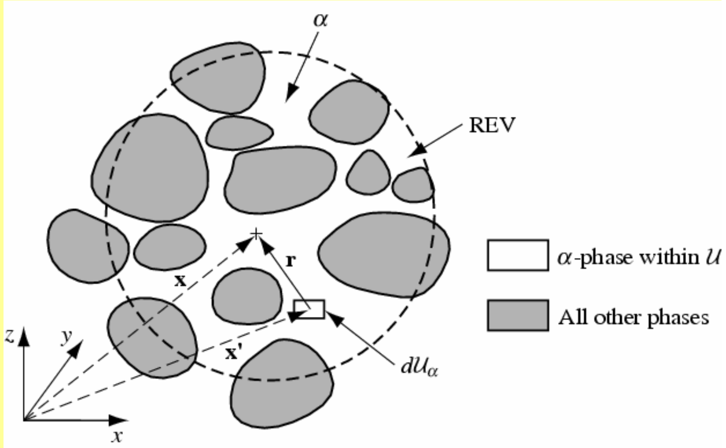
porosity, tortuosity, permeability, dispersivity

- VOLUME AVERAGING:

(Different techniques by Bear-Bachmat, by Whitaker, by Hassanizadeh-Gray..)

$$\overline{e_\alpha}^{\alpha}(\mathbf{x},t) = e_\alpha(\mathbf{x}',t;\mathbf{x}) + \overset{o}{e}_\alpha(\mathbf{x}',t;\mathbf{x})$$

Averaging rules



$$\overline{G_1 G_2}^{\alpha} = \overline{G_1 G_2}^{\alpha} + \overline{\overset{o}{G}_1 \overset{o}{G}_2}^{\alpha}$$

$$\theta_\alpha \overline{\left(\frac{\partial G}{\partial t}\right)}^{\alpha} = \frac{\partial}{\partial t}\left(\theta_\alpha \overline{G}^{\alpha}\right) - \frac{1}{\mathcal{U}_0}\int_{S_{\alpha\beta}} G\mathbf{u}\cdot\mathbf{n}\,dS$$

Note the effect of the BC's on the interface.

$$\theta_\alpha \overline{\left(\frac{\partial G_{ijk\ldots}}{\partial x_i}\right)}^{\alpha} = \frac{\partial}{\partial x_i}\left(\theta_\alpha \overline{G_{ijk\ldots}}^{\alpha}\right) + \frac{1}{\mathcal{U}_0}\int_{S_{\alpha\beta}} G_{ijk\ldots}\cdot n_i\,dS$$

Fundamental balance equation:

Starting point: the balance equation of an extensive quantity, *E*, at the microscopic level, i.e., at a point inside a phase. Intensive quantity e.

$$e = \frac{dE}{d\mathcal{U}}$$

$$\theta_\alpha = \text{volumetric fraction of } \alpha - \text{fluid phase.}$$

Source of *E*

- Differential balance of *E*:

$$\frac{\partial e_\alpha}{\partial t} = -\nabla \cdot e_\alpha \mathbf{V}_\alpha^E + \rho_\alpha \boldsymbol{\Gamma}_\alpha^E$$

Velocity of *E*

Or:

Diffusive flux of *E*

$$\frac{\partial e_\alpha}{\partial t} = -\nabla \cdot (e_\alpha \mathbf{V}_\alpha + \mathbf{j}_\alpha^E) + \rho_\alpha \boldsymbol{\Gamma}_\alpha^E, \quad \mathbf{j}_\alpha^E = e_\alpha (\mathbf{V}_\alpha^E - \mathbf{V}_\alpha)$$

Mass-averaged velocity

By averaging:

Dispersive flux of *E*

$$\frac{\partial \theta_\alpha \overline{e_\alpha}^\alpha}{\partial t} = -\nabla \cdot \theta_\alpha (\overline{e_\alpha}^\alpha \overline{\mathbf{V}_\alpha}^\alpha + \overline{\overset{o}{e}_\alpha \overset{o}{\mathbf{V}}_\alpha}^\alpha + \overline{\mathbf{j}_\alpha^E}^\alpha) - f_{\alpha\to\beta} + \theta_\alpha \overline{\rho_\alpha \boldsymbol{\Gamma}_\alpha^E}^\alpha$$

Transfer from $\alpha \to \beta$

Across $S_{\alpha\beta}$

$$f_{\alpha\to\beta} = \frac{1}{\mathcal{U}_0} \int_{S_{\alpha\beta}} \left[e(\mathbf{V}_\alpha - \mathbf{u}) + \mathbf{j}_\alpha^E \right] \cdot \mathbf{n}\, dS$$

- For mass of a fluid phase: $e \longrightarrow \rho$
- Microscopic mass balance equation

$$\frac{\partial \rho_\alpha}{\partial t} = -\nabla \cdot \rho_\alpha \mathbf{V}_\alpha$$

- The macroscopic mass balance equation for the fluid α -phase,

with $(\mathbf{V_a} - \mathbf{u}).\mathbf{n} = 0,\quad \mathbf{j_a^{m_a}} \cdot \mathbf{n} \equiv 0$ i.e. *S* is a material surface

and

$$\left| \overline{\rho_\alpha}^\alpha \overline{\mathbf{V}_\alpha}^\alpha \right| \gg \left| \overline{\overset{o}{e}_\alpha \overset{o}{\mathbf{V}}_\alpha}^\alpha \right|$$

Is:

Specific discharge

$$\frac{\partial \theta_\alpha \overline{\rho_\alpha}^\alpha}{\partial t} = -\nabla \cdot \theta_\alpha (\overline{\rho_\alpha}^\alpha \overline{\mathbf{V}_\alpha}^\alpha) \equiv -\nabla \cdot \overline{\rho_\alpha}^\alpha \mathbf{q}_\alpha \, ,\qquad \mathbf{q}_\alpha \equiv \theta_\alpha \overline{\mathbf{V}_\alpha}^\alpha$$

And for a γ-component of the α -fluid phase:

Dispersive flux of γ - mass

$$\frac{\partial \theta_\alpha \overline{c_\alpha^\gamma}^\alpha}{\partial t} = -\nabla \cdot \theta_\alpha \left(\overline{c_\alpha^\gamma}^\alpha \overline{\mathbf{V}_\alpha}^\alpha + \overline{\overset{o}{c}{}_\alpha^\gamma \overset{o}{\mathbf{V}}_\alpha}^\alpha + \overline{\mathbf{j}_\alpha^\gamma}^\alpha \right) - f_{\alpha\to\beta}^\gamma + \theta_\alpha \overline{\rho_\alpha \boldsymbol{\Gamma}_\alpha^\gamma}^\alpha ,$$

$$f_{\alpha\to\beta}^\gamma = \frac{1}{\mathcal{U}_0} \int_{S_{\alpha\beta}} \left[c_\alpha^\gamma (\mathbf{V}_\alpha - \mathbf{u}) + \mathbf{j}_\alpha^\gamma \right] \cdot \mathbf{n}_\alpha \, dS_{\alpha\beta}$$

More relevant here…..averaging the linear momentum balance equation (motion equation)

Next,

- Microscopic linear momentum balance equation for the fluid phase:

➢Microscopic linear momentum balance equation for the fluid phase:

$E \equiv \mathbf{M}, \quad e = \rho \mathbf{V}, \quad \mathbf{j}^{E} = \rho \mathbf{V}(\mathbf{V}^{M} - \mathbf{V}) \equiv -\boldsymbol{\sigma} =$ stress in the phase,

$\boldsymbol{\Gamma}^{M} \equiv \mathbf{F} =$ external body force $\left(=-g \nabla z\right)$.

$$\frac{\partial}{\partial t}\left(\rho \mathbf{V}\right) = -\nabla \cdot \left(\rho \mathbf{V}\mathbf{V} - \boldsymbol{\sigma}\right) + \rho \mathbf{F}$$

Stress =
Diffusive flux of
fluid momentum

Source

Advective flux of
fluid momentum

Next: Volume (REV) averaging

The macroscopic or averaged momentum balance equation:

$$\frac{\partial \theta_\alpha \overline{\rho_\alpha \mathbf{V}_\alpha}^{\alpha}}{\partial t} = -\nabla \cdot \theta_\alpha \left(\overline{\rho_\alpha \mathbf{V}_\alpha}^{\alpha}\, \overline{\mathbf{V}_\alpha}^{\alpha} + \overline{\overset{o}{\left(\rho_\alpha \mathbf{V}_\alpha\right)}\, \overset{o}{\mathbf{V}}_\alpha}^{\alpha} - \overline{\boldsymbol{\sigma}}^{\alpha} \right) - f^{\mathrm{M}}_{\alpha \to s} + \theta_\alpha \overline{\rho_\alpha F_\alpha}^{\alpha},$$

$$f^{\mathrm{M}}_{\alpha \to s} = \frac{1}{\mathcal{U}_0} \int_{S_{\alpha s}} \left[\rho_\alpha \mathbf{V}_\alpha (\mathbf{V}_\alpha - \mathbf{u}) - \boldsymbol{\sigma}_\alpha \right] \cdot \mathbf{n}\, dS,$$

Approximations:

Neglect dispersive momentum flux and Dispersive mass flux

$$\left| \overline{\overset{o}{\left(\rho_\alpha \mathbf{V}_\alpha\right)}\, \overset{o}{\mathbf{V}}_\alpha}^{\alpha} \right| \ll \left| \overline{\rho_\alpha \mathbf{V}_\alpha}^{\alpha}\, \overline{\mathbf{V}_\alpha}^{\alpha} \right| \simeq \left| \overline{\rho_\alpha}^{\alpha}\, \overline{\mathbf{V}_\alpha}^{\alpha}\, \overline{\mathbf{V}_\alpha}^{\alpha} \right|$$

$$\left| \overline{\overset{o}{\rho}_\alpha\, \overset{o}{\mathbf{V}}_\alpha}^{\alpha} \right| \ll \left| \overline{\rho_\alpha}^{\alpha}\, \overline{\mathbf{V}_\alpha}^{\alpha} \right|,$$

- ❑Assume: saturated flow, $\theta_\alpha \rightarrow \phi = \text{porosity}$
- ❑Solid-fluid interface is material surface with respect to fluid mass,

$$(\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} = 0$$

➢Macroscopic momentum balance equation for saturated flow

α - phase

$$\frac{\partial \phi \bar{\rho}^{\alpha} \bar{\mathbf{V}}^{\alpha}}{\partial t} = -\nabla \cdot \phi(\bar{\rho}^{\alpha} \bar{\mathbf{V}}^{\alpha} \bar{\mathbf{V}}^{\alpha} - \bar{\sigma}^{\alpha}) + \frac{1}{\mathcal{U}_0} \int_{S_{\alpha s}} \sigma \cdot \mathbf{n} \, dS + \phi \overline{\rho F}^{\alpha},$$

Or:

$$\phi \bar{\rho}^{\alpha} \frac{D \bar{\mathbf{V}}^{\alpha}}{Dt} = \nabla \cdot \phi \bar{\sigma}^{\alpha} + \frac{1}{\mathcal{U}_0} \int_{S_{\alpha s}} \sigma \cdot \mathbf{n} \, dS + \phi \overline{\rho F}^{\alpha}$$

Or:

$$\phi \bar{\rho}^{\alpha} \frac{D \bar{\mathbf{V}}^{\alpha}}{Dt} = \phi \overline{\nabla \cdot \sigma}^{\alpha} + \phi \overline{\rho F}^{\alpha},$$

with

$$\phi \bar{\rho}^{\alpha} \frac{D \bar{\mathbf{V}}^{\alpha}}{Dt} \equiv \phi \bar{\rho}^{\alpha} \left(\frac{\partial \bar{\mathbf{V}}^{\alpha}}{\partial t} + \bar{\mathbf{V}}^{\alpha} \nabla \bar{\mathbf{V}}^{\alpha} \right) = \text{material derivative}$$

$$\bar{\sigma}_{ij}^{\alpha} = \bar{\tau}_{ij}^{\alpha} - \bar{p}^{\alpha} \delta_{ij}, \qquad F_i = -g \frac{\partial z}{\partial x_i}$$

➢Hence:

- Hence:

$$\phi \overline{\rho}^{\alpha} \frac{DV_i}{Dt} = -\phi \overline{\frac{\partial p}{\partial x_i}}^{\alpha} - \phi g \overline{\rho \frac{\partial z}{\partial x_i}}^{\alpha} + \phi \overline{\frac{\partial \tau_{ij}}{\partial x_j}}^{\alpha}$$

Forces due to: pressure…….. gravity………viscous resistance to flow

Assume Newtonian fluid:

$$\tau_{ij} = \mu \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) + \lambda'' \frac{\partial V_k}{\partial x_k} \delta_{ij}$$

Bear and Bachmat (1990)….mathematical manipulations, and adding (simplifying) assumptions, e.g.,

Pressure varies monotonously within the REV, i.e., $\nabla^2 p = 0, \quad \text{inside } \mathcal{U}_{0v}$

- In the vicinity of the solid-fluid surface, the normal components of both inertial forces and viscous resistance forces are negligible, relative to those due to pressure gradient and gravity, i.e.,

$$\left| \left(\rho \frac{DV_j}{Dt} - \frac{\partial \tau_{ij}}{\partial x_i} \right) n_j \right| \ll \left| \left(\frac{\partial p}{\partial x_j} + \rho g \frac{\partial z}{\partial x_j} \right) n_j \right|$$

Assumption: $\overset{0}{\rho} \ll \overline{\rho}^{\alpha}$ within $\mathcal{U}_{0\alpha}$

↷ $$-\phi\left(\overline{\frac{\partial p}{\partial x_i}}^{\alpha} - g\rho\overline{\frac{\partial z}{\partial x_i}}^{\alpha}\right) = -\phi\left(\frac{\partial \overline{p}^{\alpha}}{\partial x_j} - g\overline{\rho}^{\alpha}\frac{\partial z}{\partial x_j}\right)T_{ji}, \quad T_{ji} = \frac{1}{\phi\mathcal{U}_0}\int_{S_{ff}} \overset{0}{x}_i n_j \, dS$$

$\mathbf{T}$ = (kind of) tortuosity

- Viscous resistance force: $\phi\overline{\frac{\partial \tau_{ij}}{\partial x_j}}^{\alpha}$ =?

Newtonian fluid: $$\tau_{ij} = \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) + \lambda''\frac{\partial V_k}{\partial x_k}\delta_{ij}$$

……..mathematical development, and assumptions about order of magnitude of terms…e.g., $\nabla\cdot\mathbf{V} = 0$ inside the fluid phase (microscopically isochoric flow)

↷ assuming $\nabla\cdot\mathbf{V} = 0$

$$\phi\overline{\frac{\partial \tau_{ij}}{\partial x_j}}^{\alpha} = \overline{\mu}^{\alpha}\left[\frac{\partial^2 q_i}{\partial x_i x_j} + \frac{\partial}{\partial x_j}\left(\frac{1}{\mathcal{U}_0}\int_{S_{fs}} V_i n_j \, dS\right) + \frac{1}{\mathcal{U}_0}\int_{S_{fs}} \frac{\partial V_i}{\partial s_n} dS\right],$$

- Important assumption: fluid sticks to the solid (“no-slip”) $\mathbf{V}\big|_{S_{sf}} = \mathbf{V}_s\big|_{S_{sf}}$

and $\mathbf{V}_{si}\big|_{S_{sf}} \approx \overline{\mathbf{V}}_{si}^{\,s}$

……………….

$$\frac{1}{\mathcal{U}_0}\int_{S_{fs}} \frac{\partial V_i}{\partial s_n}\, dS \simeq \frac{1}{\mathcal{U}_0}\int_{S_{fs}} \frac{V'_{si}\big|_{S_{sf}} - V_i\big|_{\mathbf{\Delta}}}{\mathbf{\Delta}}\, dS$$

$$\simeq \frac{1}{\mathcal{U}_0}\int_{S_{fs}} \left(\overline{V_{sj}}^{s} - \overline{V_j}^{\alpha}\right)\frac{1}{\mathbf{\Delta}_c}\left(\delta_{ij} - \widetilde{n_i n_j}^{fs}\right) dS$$

or:

e.g., hydraulic radius

$$\frac{1}{\mathcal{U}_0}\int_{S_{fs}} \frac{\partial V_i}{\partial s_n}\, dS = -\frac{C_f}{\mathbf{\Delta}_\alpha^2}\alpha_{ij}\, q_{rj}, \qquad q_{rj} = \phi\left(\overline{V_j}^{\alpha} - \overline{V_{sj}}^{s}\right)$$

$\frac{C_f}{\mathbf{\Delta}_c^2}\alpha_{ij}$ = property of the geometry of the phase.

Specific discharge, relative to the solid.

……………………..

$$\phi\frac{\overline{\partial\tau_{ij}}^{\alpha}}{\partial x_j} = \overline{\mu}^{\alpha}\left[\frac{\partial^2 q_i}{\partial x_i x_j} - \frac{C_f}{\Delta_{\alpha}^{2}}\alpha_{ij}\, q_{rj}\right]$$

Recall:

$$\phi\frac{\overline{\partial\tau_{ij}}^{\alpha}}{\partial x_j} = \frac{\partial\phi\overline{\tau}_{ij}^{\alpha}}{\partial x_j} + \frac{1}{\mathcal{U}_0}\int_{s_{\alpha s}}\tau_{ij}\cdot n_j\, dS$$

i.e., resistance force per unit volume of porous medium. Thus, the total viscous resistance is made up of:

---resistance resulting from internal friction in the fluid

---resistance expressing drag at the fluid-solid interface.

$$\frac{1}{\mathcal{U}_0}\int_{s_{\alpha s}}\tau_{ij}\cdot n_j\, dS = -\overline{\mu}^{\alpha}\frac{C_f}{\Delta_{\alpha}^{2}}\alpha_{ij}\, q_{rj}$$

$\overline{\mu}^{\alpha}\dfrac{\partial^2 q_i}{\partial x_i x_j}$ = resistance to flow due to the internal friction inside the fluid.

Finally…

$$\overline{\rho}^{\alpha}\left\{\frac{\partial q_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{q_i q_j}{\phi}\right)\right\}=-\phi\left(\frac{\partial \overline{p}^{\alpha}}{\partial x_j}+\overline{\rho}^{\alpha}g\frac{\partial z}{\partial x_j}\right)T_{ij}^{*}$$

$$+\overline{\mu}^{\alpha}\frac{\partial^2 q_{ri}}{\partial x_j x_j}-\overline{\mu}^{\alpha}\alpha_{ij}\frac{C_{\alpha}}{\Delta_{\alpha}^{2}}q_{rj}$$

T_{ij}^{*} α_{ij} , tensorial coefficients that reflect the configuration of S_{sf} (single phase saturated flow)

Special cases: $\mathbf{V}_s = 0, \quad \nabla\cdot\mathbf{q} = 0$ (isochoric flow)

$$\overline{\rho}^{\alpha}\left\{\frac{\partial q_i}{\partial t}+q_j\frac{\partial (q_i/\phi)}{\partial x_j}\right\}=-\phi\left(\frac{\partial \overline{p}^{\alpha}}{\partial x_j}+\overline{\rho}^{\alpha}g\frac{\partial z}{\partial x_j}\right)T_{ij}^{*}$$

$$+\overline{\mu}^{\alpha}\frac{\partial^2 q_i}{\partial x_j \partial x_j}-\overline{\mu}^{\alpha}\alpha_{ij}\frac{C_{\alpha}}{\Delta_{\alpha}^{2}}q_{rj}$$

➢CASE A.

Assumption:

---viscous resistance due to momentum transfer at fluid-solid interface is much larger than both inertial force and the viscous resistance to the flow inside the fluid:

$$\left|\overline{\mu}^{\alpha}\alpha_{ij}\frac{C_{\alpha}}{\mathbf{\Delta}_{\alpha}^{2}}q_{rj}\right| \gg \left|\phi\overline{\rho}^{\alpha}\frac{D\overline{\mathbf{V}}^{\alpha}}{Dt}\right|; \quad \left|\overline{\mu}^{\alpha}\alpha_{ij}\frac{C_{\alpha}}{\mathbf{\Delta}_{\alpha}^{2}}q_{rj}\right| \gg \left|\overline{\mu}^{\alpha}\frac{\partial^{2}q_{ri}}{\partial x_{j}\partial x_{j}}\right|$$

…a condition valid for $\mathrm{Re} < 1-10$

$$k_{jl} = \frac{\phi\Delta_{\alpha}^{2}}{C_{\alpha}}(\alpha_{ji})^{-1}T_{il}^{*} \qquad \mathrm{Re} = \frac{V_{c}\sqrt{k_{c}/\phi_{c}T_{c}^{*}}}{\nu_{c}}$$

$$q_{rj} \equiv \phi(\overline{V}_{j}^{\alpha} - \overline{V}_{sj}^{s}) = -\frac{k_{jl}}{\overline{\mu}^{\alpha}}\left(\frac{\partial\overline{p}^{\alpha}}{\partial x_{l}} - \overline{\rho}^{\alpha}g\frac{\partial z}{\partial x_{l}}\right)$$

i.e., Darcy's Law (saturated flow, but possibly variable density. Anisotropic soil)

➢CASE B.

<u>Assumptions:</u>

•Inertial effects are negligible

$$\left|\bar{\mu}^{\alpha}\alpha_{ij}\frac{C_{\alpha}}{\mathbf{\Delta}^{2}_{\alpha}}q_{rj}\right| \gg \left|\phi\bar{\rho}^{\alpha}\frac{D\bar{\mathbf{V}}^{\alpha}}{Dt}\right|;$$

But

•We do not neglect the effect of internal friction

i.e., $\left|\bar{\mu}^{\alpha}\frac{\partial^{2}q_{ri}}{\partial x_{j}\partial x_{j}}\right|$ is not negligible

$$q_{rk} = -\frac{k_{kj}}{\bar{\mu}^{\alpha}}\left(\frac{\partial\bar{p}^{\alpha}}{\partial x_{j}} + \bar{\rho}^{\alpha}g\frac{\partial z}{\partial x_{j}}\right) + \frac{k_{jp}(T^{*}_{pk})^{-1}}{\phi}\frac{\partial^{2}q_{rj}}{\partial x_{i}x_{i}}$$

For an isotropic porous medium:

$$q_{rj} = -\frac{k}{\bar{\mu}^{\alpha}}\left(\frac{\partial\bar{p}^{\alpha}}{\partial x_{j}} + \bar{\rho}^{\alpha}g\frac{\partial z}{\partial x_{j}}\right) + \frac{k}{\phi T^{*}}\frac{\partial^{2}q_{rj}}{\partial x_{i}x_{i}}$$

Or:

$$\mathbf{q}_r = -\frac{k}{\overline{\mu}^{\alpha}}\left(\nabla\overline{p}^{\alpha} + \overline{\rho}^{\alpha} g\nabla z\right) - \frac{k}{\phi T^{*}}\nabla^{2}\mathbf{q}_r$$

(Bachmat and Bear, 1986)

…which is, more or less, the BRINKMAN EQUATION (1948), e.g.,

$$\nabla p = \frac{\mu}{k}\mathbf{q} + \tilde{\mu}\nabla^{2}\mathbf{q},$$

$\tilde{\mu}$ = effective viscosity

…also obtained by Whitaker (1996) by averaging.

The Laplacian term is required mainly when a no-slip BC should be satisfied.

Bear and Bachmat (1990), by averaging:

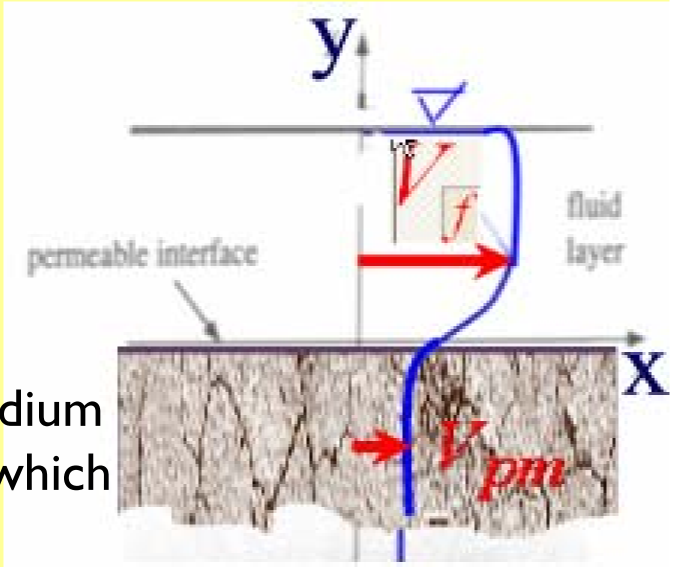
$$\frac{\tilde{\mu}}{\mu} = \frac{1}{\phi T^{*}}$$

i.e., dependent on void space geometry.

Note:

Two viscous terms: due to “friction” (momentum transfer) at the solid-fluid interface, and due to momentum transfer INSIDE the fluid, due to velocity gradient.

- Two viscous terms: due to “friction” (momentum transfer) at the solid-fluid interface, and due to momentum transfer INSIDE the fluid, due to velocity gradient.



- The Laplacian term is required mainly when a no-slip BC should be satisfied

- The Brinkman equation is used when a porous medium domain is bounded by a body of flowing fluid, along which the boundary condition can be obtained from the requirement of continuity of total normal stress, which under some simplifying assumptions concerning the viscous part of the stress, leads to *continuity of effective stress.*

- For a Newtonian fluid, the boundary condition

$$\left[\boldsymbol{\tau}_f\right]_{1,2}\cdot\mathbf{n}=0,\Rightarrow \frac{\mu}{\phi}\frac{\partial q_x}{\partial z}\bigg|_1=\mu\frac{\partial V}{\partial z}\bigg|_2\Rightarrow\frac{1}{\phi}\alpha^M\left(q_x\big|_1-q_o\right)=\frac{\partial V}{\partial z}\bigg|_2$$

$$\Rightarrow e.g., \text{B.C.}: \quad \frac{\partial V_f}{\partial z}=\frac{\alpha^M}{\sqrt{k}}\left(V_f-V_{pm}\right),$$

- Brinkman equation also used when dealing with convection in porous media.

- Another case, oscillatory flow, waves, where $\frac{\partial V}{\partial t}$ may not be neglected.

For example: for the combined fluid-solid system:

$$\phi\rho_f \frac{D_f V_{fj}}{Dt} + (1-\phi)\rho_s \frac{D_s Vsj}{Dt} = \frac{\partial \overline{\sigma_{ij}}}{\partial x_i} + \overline{\rho F_j}$$

which may serve as a starting point for various simplifications, e.g.: local acceleration much larger than the convective one, Then, for the fluid alone (in terms of the velocity relative to the solid):

$$\phi\rho_f \frac{\partial V_{ri}}{\partial t} + \phi\rho_f \frac{\partial V_{si}}{\partial t}$$

$$= -\phi\left(\frac{\partial p}{\partial x_j} + \rho_f g \frac{\partial z}{\partial x_j}\right) T^*_{ji} - \phi^2 \mu T^*_{ij} \left(k_{jl}\right)^{-1} V_{rl}$$

Other **micro to macro** transformation techniques (= upscaling) are also available, e.g.:

- Volume averaging technique with closure (e.g., Whitaker, 1999)
(we always need some kind of closure hypothesis of a macroscopic or microscopic nature, which expresses the unknown coupling between solid and fluid phases…..eventually takes the form of a macroscopic coefficient (e.g., permeability).

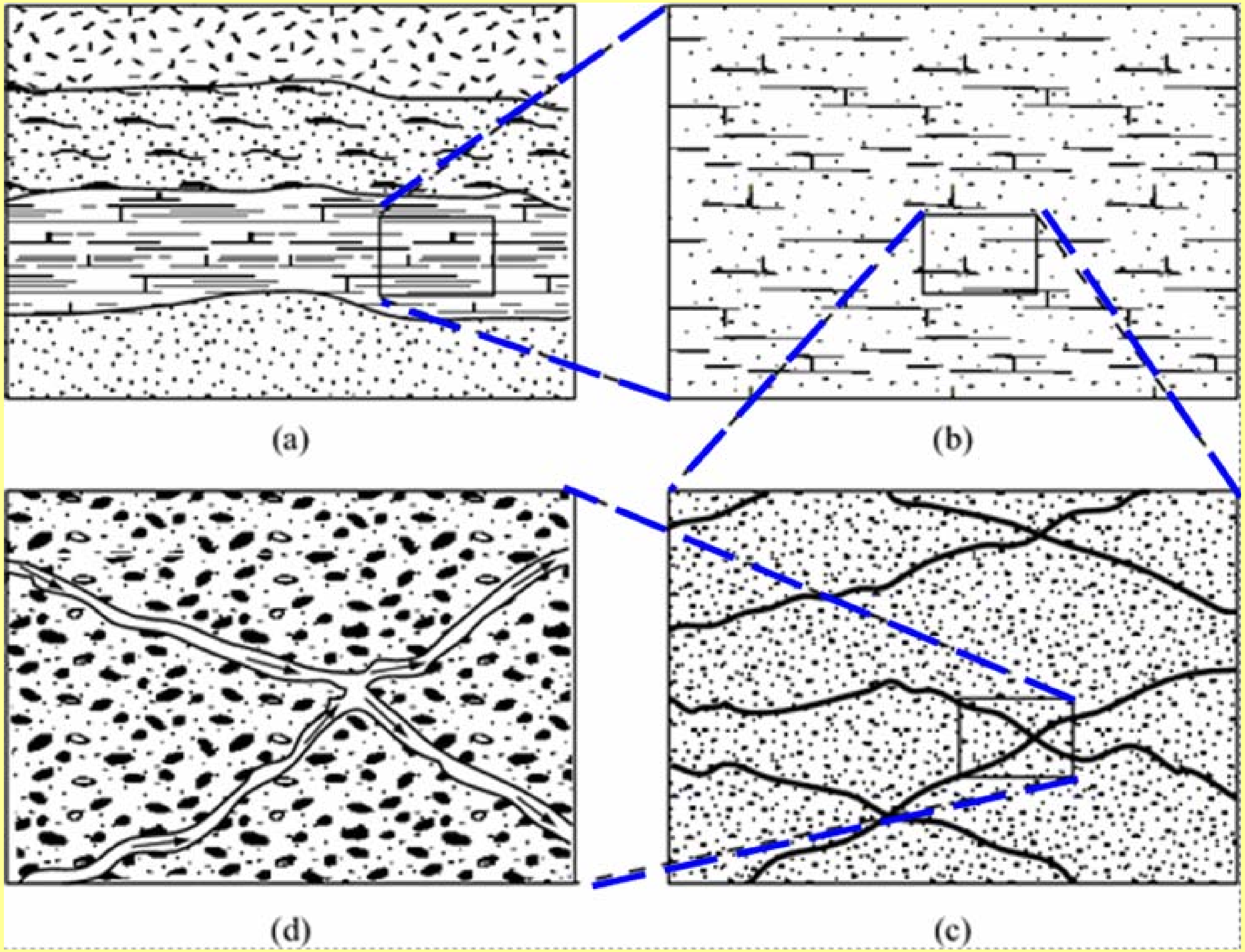
- In an excellent series of papers, Majid Hassanizadeh and Bill Gray, presented a complete and comprehensive approach to the averaging of all phenomena of transport in porous media, constitutive relations, etc.

Additional approaches to macroscopization…

- MIXTURE THEORY (especially in connection with deformable porous medium)

- ASYMPTOTIC HOMOGENIZATION method (or formal asymptotic expansion, or double-scale asymptotic expansion)

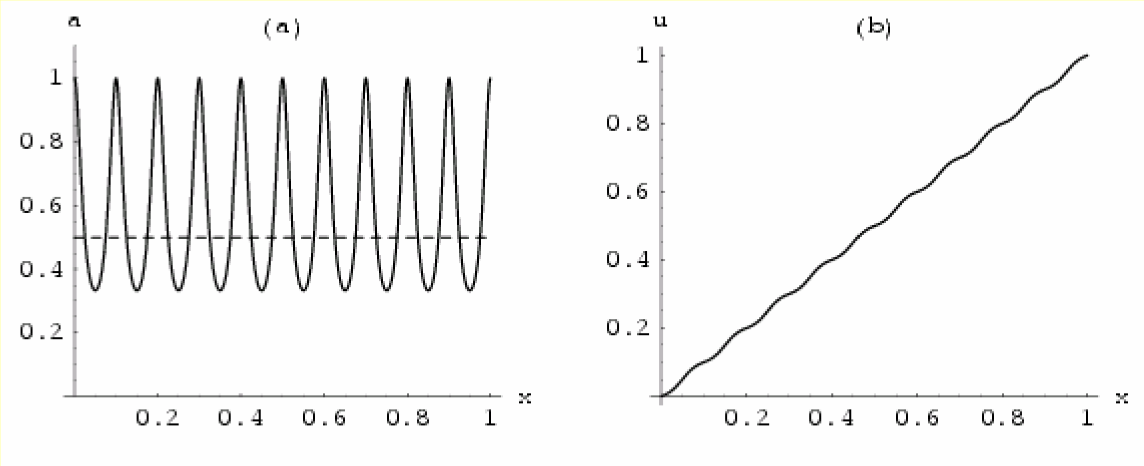
- So far, two scales: microscopic ⟶ macroscopic
- Multiple scales



- Or a layered porous medium.

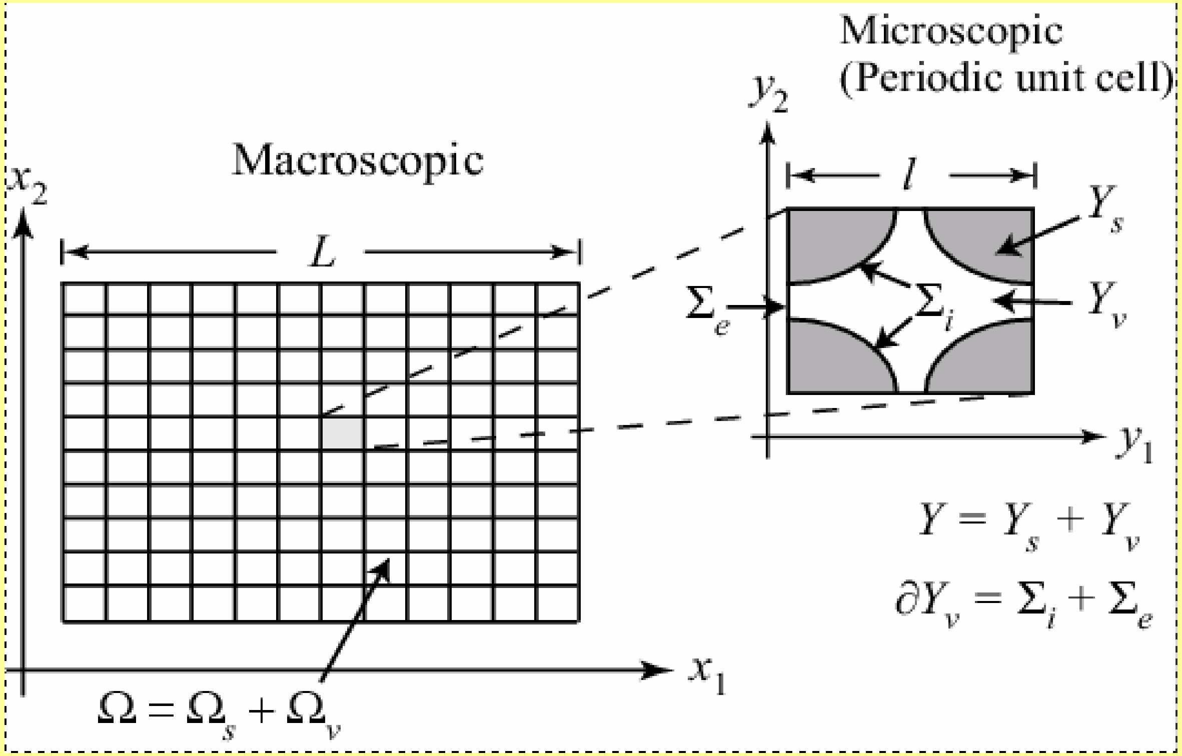
Homogenization of Second-Order Equation

The homogenization method is a mathematical method used for the study of the initial-value and boundary-value problems in domains with a periodic structure. If the dimensions of a period are small compared to the size of the whole domain, it is natural to define a small parameter ε as the ratio between those dimensions. The study of the initial-value and boundary-value problems is associated with this small parameter. The asymptotic process $\varepsilon \to 0$ represents the transition from microscopic macroscopic phenomena.



(Ene & Polisevski, 1987)

➢ASYMPTOTIC HOMOGENIZATION method



- ➢Assumption: The medium at the macroscopic level exhibits a periodic structure, based on a basic cell at the microscopic level (i.e., existence of scale separation).

- ➢The transformation from micro to macro level is derived by making use of the small parameter ,

- Objective: Derive Darcy's law by the method of asymptotic homogenization (e.g., Ene and Polisevski, 1987; Mei and Auriault, 1989, among many others)

- Starting point: The balance equation of linear momentum

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) = -\nabla \cdot (\rho \mathbf{V}\mathbf{V} - \boldsymbol{\sigma}) + \rho \mathbf{F}$$

Body force due
To gravity:

$$\mathbf{F} = -g\nabla z$$

- The fluid: Newtonian, compressible

$$\tau = \mu\left(\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right) + \lambda(\nabla\cdot\mathbf{V})\mathbf{I}, \qquad \sigma = \tau - p\mathbf{I}$$

- Simplified form of the momentum balance equation (steady state; non-deformable):

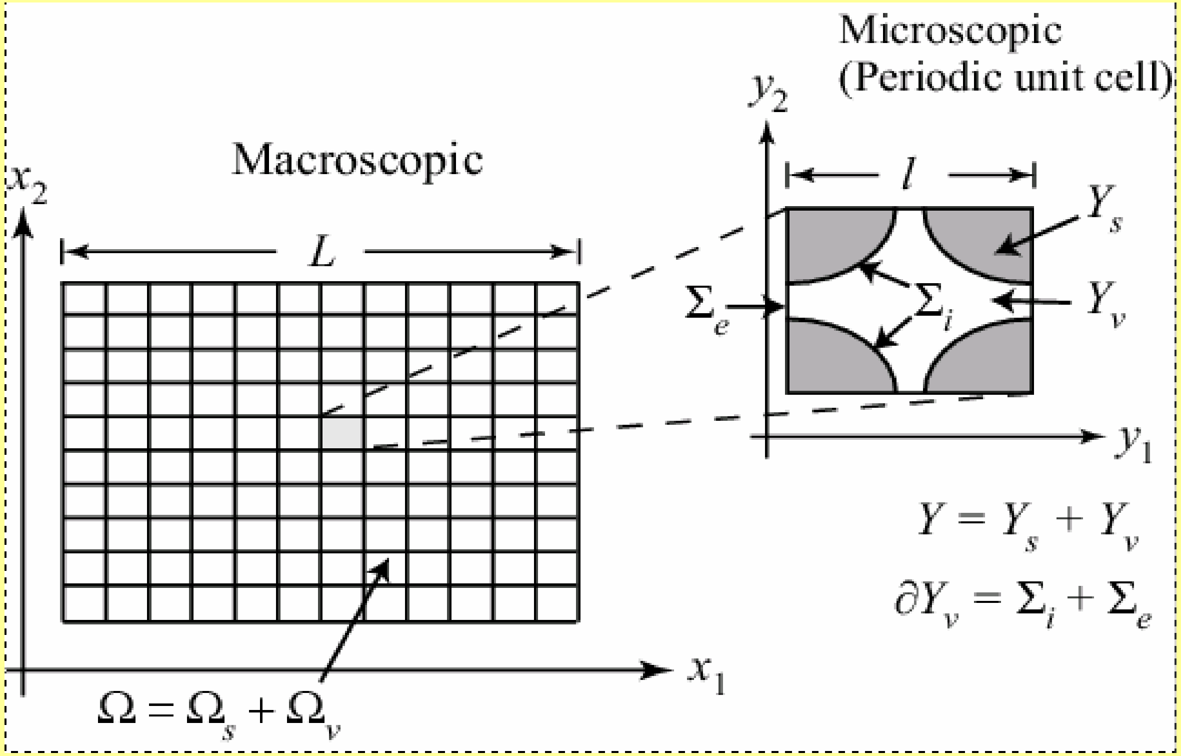
$$\nabla\cdot(\rho\mathbf{V}\mathbf{V} - \boldsymbol{\sigma}) + \rho\mathbf{F} = 0,$$

or:

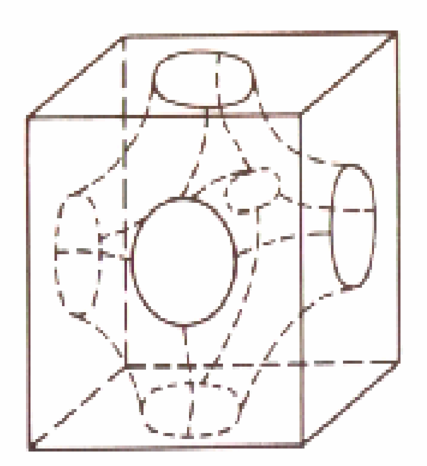
$$\rho(\mathbf{x})\big(\mathbf{v}(\mathbf{x})\cdot\nabla\big)\mathbf{v}(\mathbf{x}) = -\nabla p(\mathbf{x}) + \mu\nabla^2\mathbf{v}(\mathbf{x}) + (\lambda+\mu)\nabla\big(\nabla\cdot\mathbf{v}(\mathbf{x})\big) + \rho(\mathbf{x})\mathbf{g},$$

$\mathbf{x}$….macroscopic coordinates.

➢ASYMPTOTIC HOMOGENIZATION method
(or formal asymptotic expansion,
or double-scale asymptotic expansion)



In 3-d

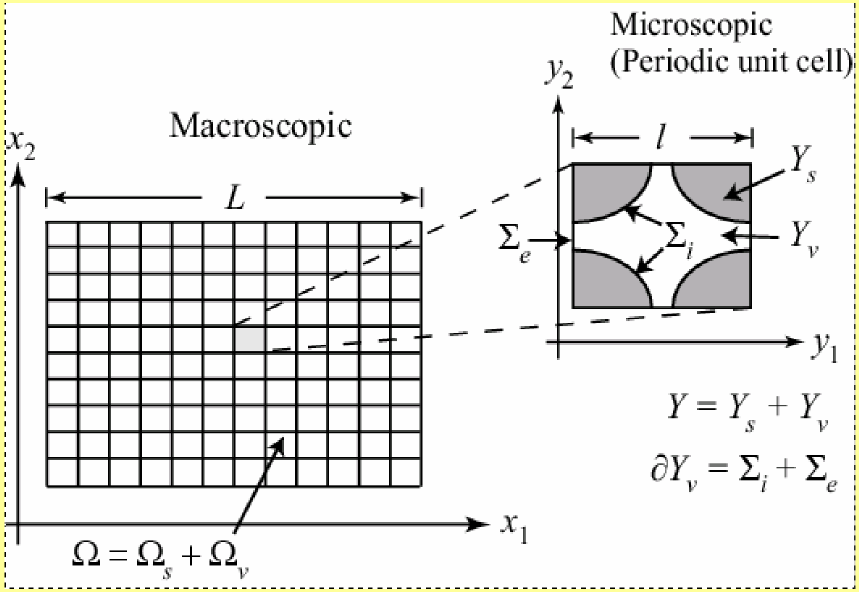


(Ene & Polisevski, 1987)

- ➢Assumption: The medium at the macroscopic level exhibits a periodic structure, based on a basic cell at the microscopic level (i.e., existence of scale separation).

- ➢The transformation from micro to macro level is derived by making use of the small parameter ,

➢Darcy's law by homogenization



- ❖Porous medium $\Omega = \Omega_v + \Omega_s$ length L
- ❖Void-solid interface Γ_i
- ❖Domain visualized as a periodic structure with elementary cells, $Y = Y_s + Y_v$
- ❖Two scales: L and l .
- ❖There exists a perturbation parameter:

$$\varepsilon = \frac{l}{L} = \frac{\text{size of basic cell}}{\text{characteristic length of domain}}, \quad l \ll L, \quad \varepsilon \ll 1.$$

- ❖Local system, $\mathbf{y}$. $\longrightarrow$ $\mathbf{y} = \dfrac{\mathbf{x}}{\varepsilon}$
- ❖The transformation from micro to macro level is derived by making use of a small parameter. Such a parameter is: ε

- Again, the momentum balance equation in a fluid continuum:

$$\rho^{\varepsilon}(\mathbf{x},\varepsilon)\left(\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)\cdot\nabla\right)\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)=-\nabla p^{\varepsilon}(\mathbf{x},\varepsilon)+\mu\nabla^{2}\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)$$

$$+(\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)\right)+\rho^{\varepsilon}(\mathbf{x},\varepsilon)\mathbf{g}.$$

- Add, no-slip B.C.: $\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)=0, \quad \mathbf{x}\in\mathbf{\Gamma}_{i}$ (at the interface with the solid)

- We need also the mass balance equation (steady state) in a fluid continuum:

$$\nabla\cdot\left[\rho^{\varepsilon}(\mathbf{x},\varepsilon)\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)\right]=0, \qquad \mathbf{x}\in\mathbf{\Omega}_{v}$$

- Constitutive relation for the fluid, e.g.:

$$\rho^{\varepsilon}=\rho^{\varepsilon}(p^{\varepsilon})\Rightarrow\rho^{\varepsilon}(p^{\varepsilon})=\rho^{\varepsilon}(p^{o})\left[1+\beta\left(p^{\varepsilon}-p^{o}\right)\right],$$

- Assumption: each dependent variable can be expanded into an asymptotic series in ε :

y…microscopic scale

$$p^{\varepsilon}(\mathbf{x},\varepsilon)=p_{o}(\mathbf{x},\mathbf{y})+\varepsilon p_{1}(\mathbf{x},\mathbf{y})+\varepsilon^{2}p_{2}(\mathbf{x},\mathbf{y})+\ldots,$$

$$\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon)=\mathbf{V}_{o}(\mathbf{x},\mathbf{y})+\varepsilon\mathbf{V}_{1}(\mathbf{x},\mathbf{y})+\varepsilon^{2}\mathbf{V}_{2}(\mathbf{x},\mathbf{y})+\ldots,$$

$$\rho^{\varepsilon}(\mathbf{x},\varepsilon)=\rho_{o}(\mathbf{x},\mathbf{y})+\varepsilon\rho_{1}(\mathbf{x},\mathbf{y})+\varepsilon^{2}\rho_{2}(\mathbf{x},\mathbf{y})+\ldots,$$

Again:

$$p^{\varepsilon}(\mathbf{x},\varepsilon) = p_o(\mathbf{x},\mathbf{y}) + \varepsilon p_1(\mathbf{x},\mathbf{y}) + \varepsilon^2 p_2(\mathbf{x},\mathbf{y}) + \ldots,$$

$$\mathbf{V}^{\varepsilon}(\mathbf{x},\varepsilon) = \mathbf{V}_o(\mathbf{x},\mathbf{y}) + \varepsilon \mathbf{V}_1(\mathbf{x},\mathbf{y}) + \varepsilon^2 \mathbf{V}_2(\mathbf{x},\mathbf{y}) + \ldots,$$

$$\rho^{\varepsilon}(\mathbf{x},\varepsilon) = \rho_o(\mathbf{x},\mathbf{y}) + \varepsilon \rho_1(\mathbf{x},\mathbf{y}) + \varepsilon^2 \rho_2(\mathbf{x},\mathbf{y}) + \ldots,$$

Note: $\mathbf{y} \in Y_v$, $\mathbf{x} \in \Omega$, i.e., we have extended the existence of the variable to the entire domain.

However, since $\mathbf{x}$ really corresponds to points only in the fluid continuum, i.e., in Ω_v , we have to multiply the variables by the characteristic function $\gamma(\mathbf{x})$, (=1 in Ω_v , =0 in Ω_s)

- By inserting the perturbed expressions into the equation of state:

$$\rho_0 = \rho\left(p^0\right)\left[1 + \beta(p_0 - p^0)\right]; \quad \rho_1 = \rho\left(p^0\right)\beta p_1; \quad \rho_2 = \rho\left(p^0\right)\beta p_2; \quad (a)$$

- We use the two-scale differentiation:

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y; \quad \nabla_x = \frac{\partial}{\partial x_i}\mathbf{i} + \frac{\partial}{\partial x_y}\mathbf{j} + \frac{\partial}{\partial x_z}\mathbf{k}$$

- Mathematical development and comparison of terms of the same order of ε

- From the N-S equation:

$$O(\varepsilon^{-1}): \qquad \nabla_y p_0(\mathbf{x},\mathbf{y}) = 0, \qquad \mathbf{x} \in \Omega_v, \quad \mathbf{y} \in Y_v, \qquad (b)$$

$$O(\varepsilon^{0}): \qquad \mu\nabla_y^2 \mathbf{V}_0(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_y\left[\nabla_y \cdot \mathbf{V}_0(\mathbf{x},\mathbf{y})\right]$$

$$= \nabla_y p_1(\mathbf{x},\mathbf{y}) + \nabla_x p_0(\mathbf{x},\mathbf{y}) - \rho_0(\mathbf{x},\mathbf{y})\mathbf{g},$$

- From the mass balance equation:

$$O(\varepsilon^{1}): \qquad \nabla_y \cdot \left[\rho_0(\mathbf{x},\mathbf{y})\mathbf{V}_0(\mathbf{x},\mathbf{y})\right] = 0, \qquad (c)$$

$$O(\varepsilon^{2}): \qquad \nabla_x \cdot \left[\rho_0(\mathbf{x},\mathbf{y})\mathbf{V}_0(\mathbf{x},\mathbf{y})\right] + \nabla_y \cdot \left[\rho_0(\mathbf{x},\mathbf{y})\mathbf{V}_1(\mathbf{x},\mathbf{y})\right]$$

$$\nabla_y \cdot \left[\rho_1(\mathbf{x},\mathbf{y}) \cdot \mathbf{V}_0(\mathbf{x},\mathbf{y})\right] = 0, \quad (d)$$

- From the boundary conditions:

$$\mathbf{V}_0(\mathbf{x},\mathbf{y}) = \mathbf{V}_1(\mathbf{x},\mathbf{y}) = \mathbf{V}_2(\mathbf{x},\mathbf{y}) = \ldots 0, \qquad \mathbf{x} \in \Gamma_i, \quad \mathbf{y} \in \Sigma_i,$$

From (b):

$$p_0(\mathbf{x},\mathbf{y}) = p_0(\mathbf{x})$$

From (a):

$$\rho_0(\mathbf{x},\mathbf{y}) = \rho_0(\mathbf{x})$$

- From $O(\varepsilon^1)$ balance equation (c):

$$\nabla_y \cdot \left[\mathbf{V}_0(\mathbf{x}, \mathbf{y}) \right] = 0,$$

- From $O(\varepsilon^2)$ mass balance equation (d):

$$\nabla_x \cdot \left[\rho_0(\mathbf{x}) \mathbf{V}_0(\mathbf{x}, \mathbf{y}) \right] + \nabla_y \cdot \left[\rho_0(\mathbf{x}) \mathbf{V}_1(\mathbf{x}, \mathbf{y}) \right] + \nabla_y \cdot \left[\rho_1(\mathbf{x}, \mathbf{y}) \mathbf{V}_o(\mathbf{x}, \mathbf{y}) \right] = 0,$$

…the $O(\varepsilon^0)$ Navier-Stokes equation:

$$\mu \nabla_y^2 \mathbf{V}_0(\mathbf{x}, \mathbf{y}) = \nabla_y p_1(\mathbf{x}, \mathbf{y}) + \nabla_x p_0(\mathbf{x}) - \rho_0(\mathbf{x}) \mathbf{g},$$

- These two equations have to be averaged over the Y -cell, in order to smooth out the microscopic variations and to produce equations in terms of the macroscopic, ***x***-coordinates only.

- **Averaging is achieved by integration.**

- These last 2 equations: mass and momentums balance equation. have to be averaged over the Y -cell, in order to smooth out the microscopic variations and to produce equations in terms of the macroscopic, ***x***-coordinates only.

- Averaging …..by integration:
  - For the velocity $\mathbf{V}_o$:

$$\mathbf{q}_0(\mathbf{x}) = \frac{1}{|Y|}\int_{Y_v} \mathbf{V}_0(\mathbf{x},\mathbf{y})\, d\mathcal{U}_v(\mathbf{y})$$

  - “Homogenized velocity” = specific discharge

Volume of ***Y***—the entire cell

- By averaging the $O(\varepsilon^2)$: mass balance equations:

$$\nabla_x \cdot \left[\rho_o(\mathbf{x})\mathbf{q}_o(\mathbf{x})\right] = 0$$

i.e., the macroscopic mass balance equation.

- In a similar way, for the momentum balance equation (Navier-Stokes), we lead to the solution:

2nd rank tensor

$$\mathbf{V}_0(\mathbf{x},\mathbf{y}) = -\frac{1}{\mu}\mathbf{w}(\mathbf{y})\cdot\left[\nabla_x p_0(\mathbf{x}) - \rho_0(\mathbf{x})\mathbf{g}\right]$$

…and by integration:

$$\boxed{\mathbf{q}_0(\mathbf{x}) = -\frac{\mathbf{k}}{\mu}\cdot\left[\nabla_x p_0(\mathbf{x}) - \rho_0(\mathbf{x})\mathbf{g}\right],} \quad \mathbf{g} = -g\nabla_x z$$

DARCY's LAW

in which

$$\mathbf{k}(\mathbf{x}) = \frac{1}{|Y|}\int_{Y_v}\mathbf{w}(\mathbf{y})\,d\mathcal{U}_v(\mathbf{y})$$

is the (intrinsic) permeability.

- The macroscopic driving force (per unit volume of fluid) :

$$-\left(\nabla p_0 + +\rho_0 g\nabla z\right)$$

- We have thus shown that Darcy's law is simplified form of the N-S equation.

again

- Some conclusions from the above development:
  - ☐ **k** is a property of the pore geometry only (and so is **w** .)
  - ☐ **w**, and hence **k**, is a second rank symmetric tensor.
  - ☐ **k** is positive definite.

- Recall: we have assumed

(1) Periodicity, and

(2) Separation of scales, with

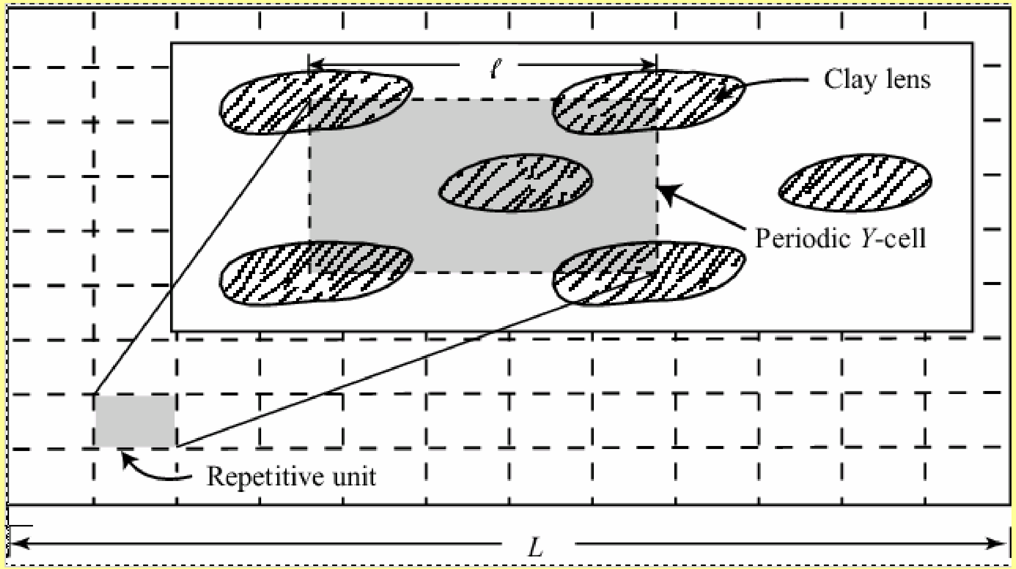
- Larger scale; field scale.

Fields are highly heterogeneous, e.g., with respect to ***K***.

We are not interested in the details (anyway, we do not really know ***K(x)***.

- In a similar way, we can apply the homogenization method to transform a Flow model from the macroscopic scale to the megascopic (= field) scale.

$$\varepsilon = \frac{l}{L} \ll 1$$



- Objective: macroscopic ⟶ megascopic (field) scale.

- For example, we may consider the balance equation (macroscopic level):

$$\nabla \cdot \mathbf{q}(\mathbf{x}) = 0,$$

or:

$$\nabla \cdot \left[\mathbf{K}(\mathbf{x}) \cdot \nabla h(\mathbf{x})\right] = 0, \quad \mathbf{x} \in \Omega$$

$$h(\mathbf{x}) = h_D(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$

- Following a similar procedure, by inserting

$$h^{\varepsilon}(\mathbf{x},\varepsilon)=h_o(\mathbf{x},\mathbf{y})+\varepsilon h_1(\mathbf{x},\mathbf{y})+\varepsilon^2 h_2(\mathbf{x},\mathbf{y})+\ldots,$$

into the continuity equation and boundary conditions, and assembling terms of the same order of ε , we obtain from the $O(\varepsilon^{-2})$ terms:

$$\nabla_y\cdot\left[\mathbf{K}(\mathbf{y})\cdot\nabla_y h_0(\mathbf{x},\mathbf{y})\right]=0,\quad \mathbf{x}\in\Omega,\quad \mathbf{y}\in Y$$

etc. Eventually, we obtain the macroscopic mass balance equation:

$$\nabla_{\mathbf{x}}\cdot\left[\mathbf{K}^{eq}\cdot\nabla_x h_0(\mathbf{x})\right]=0,\quad \mathbf{x}\in\Omega,$$

where $\mathbf{K}^{eq}$ is the effective/equivalent hydraulic conductivity, defined by

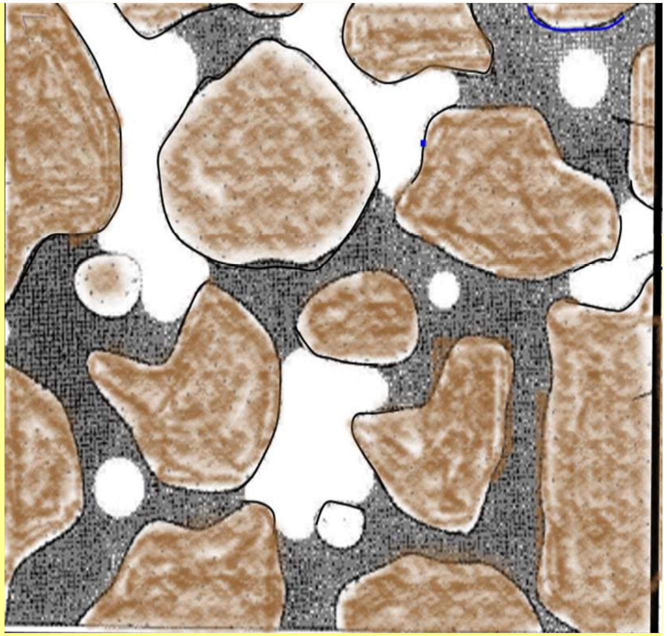
$$\mathbf{K}^{eq}=\frac{1}{|Y|}\int_Y \mathbf{K}(\mathbf{y})d\mathcal{U}(\mathbf{y})+\frac{1}{|Y|}\int_Y \mathbf{K}(\mathbf{y})\cdot\nabla_y\mathbf{w}(\mathbf{y})d\mathcal{U}(\mathbf{y})$$

can be shown to be a symmetric, positive definite, second rank tensor.

We have thus extended Darcy's law and ***K*** to various scales.

➢UNSATURATED / MULTIPHASE FLOW

To honor Darcy, …we still refer to the motion equation for a phase in multiphase flow as DARCY's Law.



- ➢The subject has been of interest to soil physicists (water in the vadose zone) and to reservoir engineers ….OIL, WATER, GAS in reservoirs.
Nowadays, it is of interest also to hydro-geologists, because of subsurface pollution.

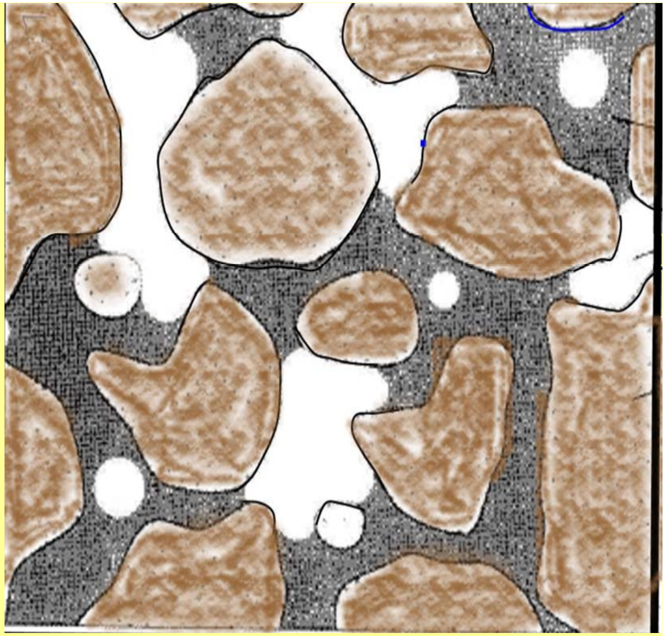
- ➢The simplest approach is to assume that each fluid phase maintains a network of passages: the wetting fluid in the larger pores, with "friction" between fluid and solid, etc. This enables an extension of Darcy's law to each phase, with the permeability a function of the saturation..

- ➢However, one could also realize that the situation---even in the case of water and air—is (much) more complicated, e.g., the presence of liquid vapor in the gas, the presence of a wetting fluid film on the solid, and the presence of liquid-gas interfaces. Also, at low saturation, surface forces may play a significant role.

Recall: we focus only on the motion equation.

- Also, because the subject was investigated by soil physicists, they felt that they have to take into account the fact that in the subsurface, it is not “pure water”, but a “soil solution”, with dissolved matter.

- Thus, perhaps it is more than just a simple extension of the saturated Darcy law.



Some pioneering works:

- Buckingham (1907) was, probably among the first to analyze unsaturated flow. He introduced the concept of CAPILLARY POTENTIAL, Ψ to describe the movement of moisture in soil.

- Additional contributions: …..Gardner (1920)……Richard (1931) extended Darcy’s law to unsaturated flow.

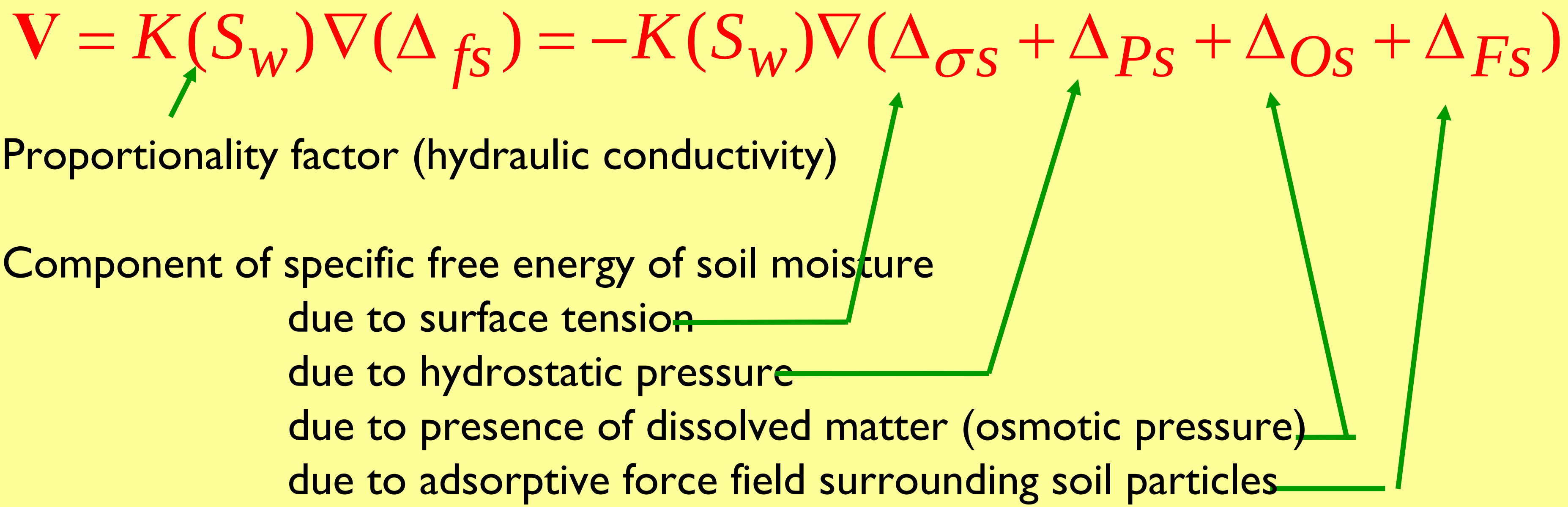
$$\mathbf{q}_w = -K'(S_w)\nabla(\Psi + gz), \quad \Psi = \int_{p_o}^{p_w} \frac{dp}{\rho(p_w)}$$

- Childs (1936) also extended the saturated flow equation to flow of water in the unsaturated zone.

No attempt to present here a historical review…but

- We should mention the most important work of Edelfson and Anderson (HILGARDIA, 1943) who approached the problem from the point of view of the principles of thermodynamics. Gravity and capillarity are not sufficient to account for the very high value of Ψ at low saturations. They postulated the concept of adsorptive forces, to explain the thin water films on the solid

They suggested:



$$\mathbf{V} = K(S_w)\nabla(\Delta_{fs}) = -K(S_w)\nabla(\Delta_{\sigma s} + \Delta_{Ps} + \Delta_{Os} + \Delta_{Fs})$$

Proportionality factor (hydraulic conductivity)

Component of specific free energy of soil moisture
due to surface tension
due to hydrostatic pressure
due to presence of dissolved matter (osmotic pressure)
due to adsorptive force field surrounding soil particles

- They suggested that Ψ is Gibb's free energy of the water in the soil. (neglecting kinetic energy)

- Water in the unsaturated zone is usually not pure water, and not under isothermal conditions. This requires that we consider the issue of the total energy in the water, or the total potential in the water, regarded as a solution. (soil-solution).

- We have to take into account BOTH fluid-fluid interface (…capillary pressure…) and fluid-solid interfaces, also when water takes the form of thin films.

- So, what is the driving force in the unsaturated zone? A gradient of what?

- Day (1942), identified Edelfson and Anderson's Ψ as

Partial Gibbs molar free energy (chemical potential) of water.

$$\mu^{\gamma} = \left.\frac{\partial G}{\partial N^{w}}\right|_{p.T,N^{\gamma}} \quad \Rightarrow \quad \mathbf{q} = -k \cdot \nabla \mu^{\gamma}$$

Water as a component..

POTENTIALS ?

- Thus, we have to go to the fundamentals of thermodynamics in order to understand what is a POTENTIAL, kinds of potential etc., and how is the TOTAL POTENTIAL of water, as a component in the soil solution, related to the CHEMICAL POTENTIAL.
- **Unlike** Darcy's original work, we are dealing now also with nonisothermal flow in the presence of concentration gradients:

- What "potential" should be used in Darcy's law for unsaturated flow?

  - Should the gradient of the free energy of the aqueous phase be used in Darcy's Law, or that of the water component?

  - Is there a difference between the diffusive flux of water as a component and the advective flux of the aqueous phase ?

  - Is the potential defined as free energy identical to the matric potential ?

….and make sure that everything is properly transformed from the microscopic to the macroscopic level…the answers are important for nonisothermal, unsaturated flow with dissolved matter, at low saturations.

- In physics: “potential” at a point is defined as the work that is required in order to transfer a unit mass from a given reference position to that point.

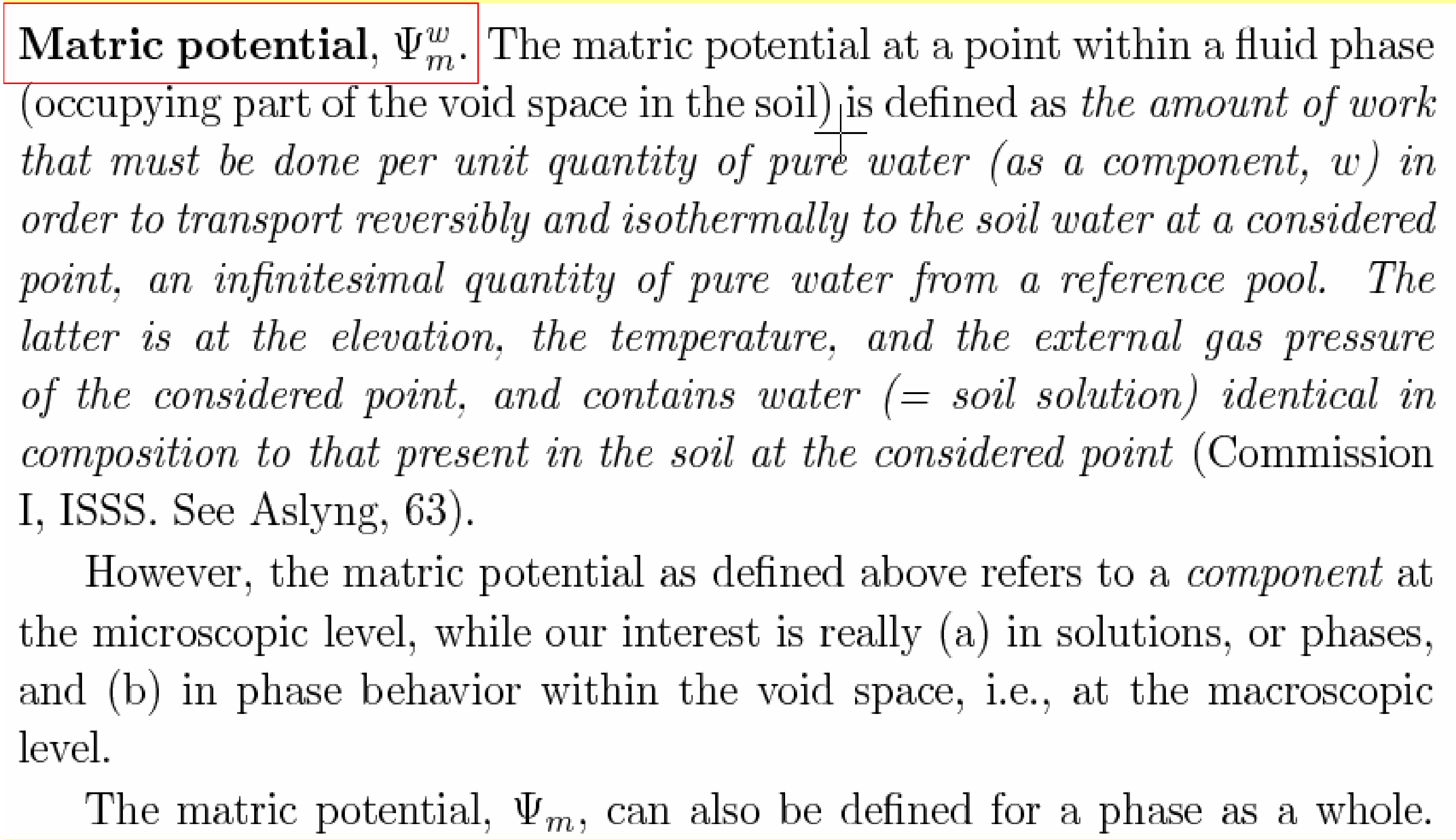
- What potential (or, potentials) should be used for unsaturated flow?

- **Some thoughts and definitions about potentials associated with Phenomena of transport in porous media.**

Total potential, Ψ^w_{total}. This potential of the water component, H_2O, at a point in the soil is defined by Commission I of the International Soil Science Society (Aslyng, 1963) as

> the amount of work that must be done per unit quantity of pure water in order to transport reversibly and isothermally to the soil water at a considered point, an infinitesimal quantity of pure water from a pool that contains pure water. The pool is at a specified elevation, and with the same temperature and external gas pressure as at the considered point.

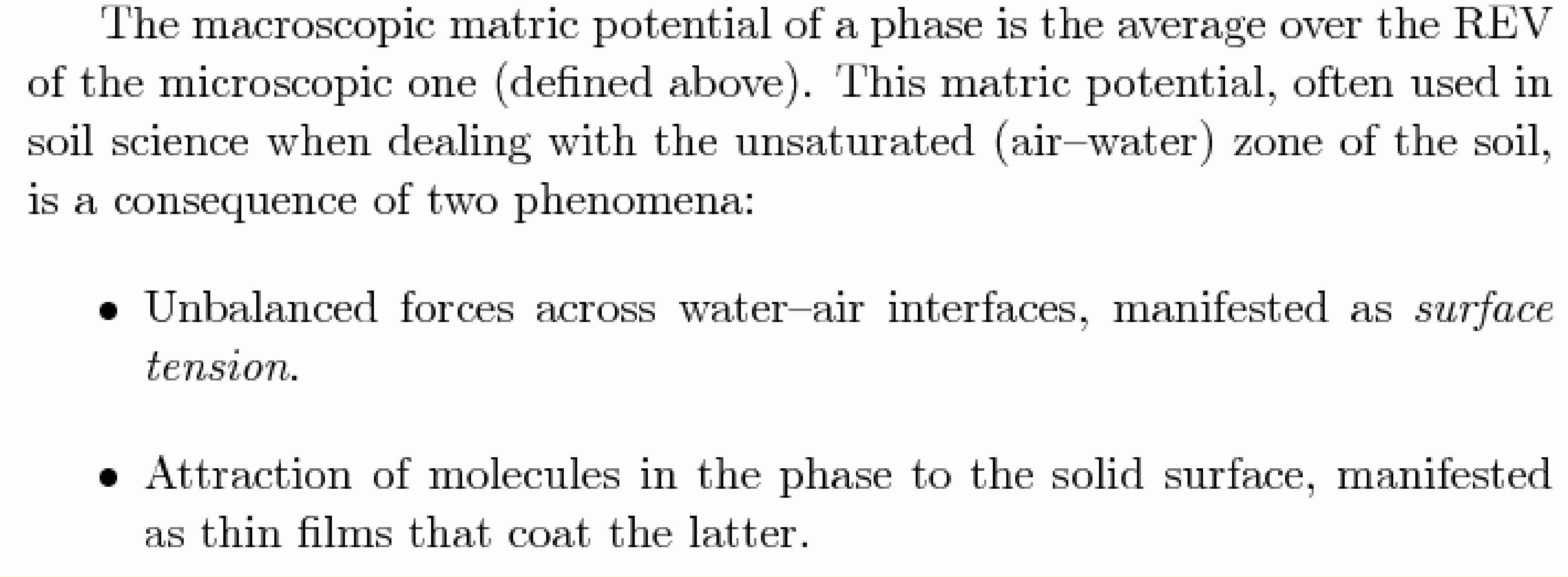
Here 'pure water' refers to the water (H_2O) as a *substance*, or a *component*, while the 'soil water' is the liquid phase (or soil solution) in the unsaturated zone, which contains both water and dissolved matter as components.



Matric potential, Ψ_m^w. The matric potential at a point within a fluid phase (occupying part of the void space in the soil) is defined as *the amount of work that must be done per unit quantity of pure water (as a component, w) in order to transport reversibly and isothermally to the soil water at a considered point, an infinitesimal quantity of pure water from a reference pool. The latter is at the elevation, the temperature, and the external gas pressure of the considered point, and contains water (= soil solution) identical in composition to that present in the soil at the considered point* (Commission I, ISSS. See Aslyng, 63).

However, the matric potential as defined above refers to a *component* at the microscopic level, while our interest is really (a) in solutions, or phases, and (b) in phase behavior within the void space, i.e., at the macroscopic level.

The matric potential, Ψ_m, can also be defined for a phase as a whole.



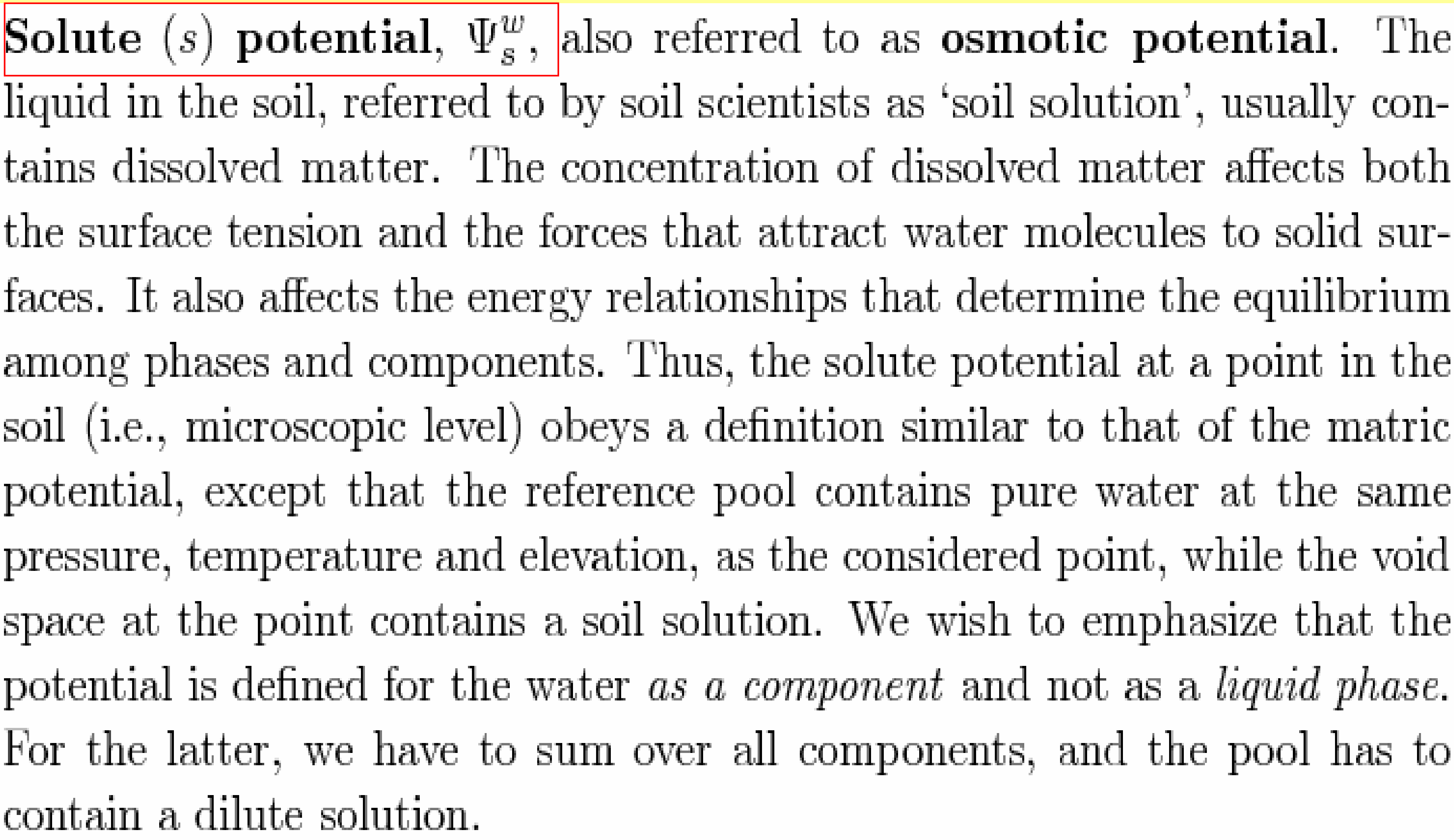
The macroscopic matric potential of a phase is the average over the REV of the microscopic one (defined above). This matric potential, often used in soil science when dealing with the unsaturated (air–water) zone of the soil, is a consequence of two phenomena:

- Unbalanced forces across water–air interfaces, manifested as *surface tension*.
- Attraction of molecules in the phase to the solid surface, manifested as thin films that coat the latter.

The definition of matric potential incorporates also the effect of the attractive forces acting upon these films. Nitao and Bear (96) showed that

$$\Psi_m = -\frac{\widetilde{2\gamma_{\ell g}}^{\ell g}}{r^*} + \widetilde{\varphi}^{\ell g},$$

where $\widetilde{A}^{\ell g}$ denotes an average of A over the $\ell - g$-interface, and φ denotes the *surface potential* to be discussed below. The matric potential is, thus, not identical to the *capillary potential*, which is associated with capillary forces only. The surface potential at the $\ell - g$-interface becomes negligible at high saturations as the distance from the interface to the solid surface increases. Consequently, the effect of the adsorbed water films becomes negligible. The matric potential is, then, essentially equal to the capillary potential. If we define the matric potential *per unit volume of water*, the matric potential in this range of saturations is identical to the capillary pressure.



Solute (s) **potential**, Ψ_s^w, also referred to as **osmotic potential**. The liquid in the soil, referred to by soil scientists as ‘soil solution’, usually contains dissolved matter. The concentration of dissolved matter affects both the surface tension and the forces that attract water molecules to solid surfaces. It also affects the energy relationships that determine the equilibrium among phases and components. Thus, the solute potential at a point in the soil (i.e., microscopic level) obeys a definition similar to that of the matric potential, except that the reference pool contains pure water at the same pressure, temperature and elevation, as the considered point, while the void space at the point contains a soil solution. We wish to emphasize that the potential is defined for the water *as a component* and not as a *liquid phase*. For the latter, we have to sum over all components, and the pool has to contain a dilute solution.

Soil water (sw) potential, Ψ^w_{sw}. This potential combines the work required to overcome the forces due to both surface forces, pressure, and concentration differences between the reference reservoir and the considered point. The soil-water potential is, thus, the sum of the matric and the solute potentials.

Taking into account the effect of dissolved matter, and following a derivation similar to that for Kelvin's equation, (3.1.29), we may write:

$$\Psi^w_{sw} = -\frac{\rho^w_w RT}{M^w} \ln \frac{p^v}{p^{v_o}}, \tag{3.1.40}$$

in which Ψ^w_{sw} is per unit volume of water in the void space. Note that superscript w refers to water as a component, while subscript w refers to

Gravity potential, Ψ_g^w. This potential expresses the change in the potential energy associated with the elevation of the considered point above the reference reservoir. Thus, we can use the same definition as that of the matric potential, except that the considered point and the reference reservoir are at different elevations, and both reservoirs have flat interfaces. When expressed per unit weight, the gravity potential is equal to the elevation, $z - z_0$, of the point. When expressed per unit volume, the gravity potential for the w-component is given by $\rho_w^w g(z - z_0)$, assuming that ρ_w^w does not change appreciably from z_0 to z.

Thermal potential, Ψ_T^w. This potential expresses the change in the free energy associated with the temperature of the considered point above that prevailing in the reference reservoir. Thus, we can use the same definition as that of the matric potential, except that the considered point and the reference pure water reservoir are at different temperatures.

Total potential, Ψ_{total}^w. The total potential for a w-component, may now be defined as:

$$\boxed{\Psi_{total}^w = \Psi_m^w + \Psi_s^w + \Psi_g^w + \Psi_T^w = \Psi_{sw}^w + \Psi_g^w + \Psi_T^w.} \qquad (3.1.41)$$

Back to flow in the unsaturated zone (water-air, wetting-nonwetting fluids)

- By averaging, say of the Navier-Stokes equations:

$$\mathbf{q}_{rw} = -\frac{\mathbf{k}_w^w(S_w)}{\mu_w}\cdot\left(\nabla p_w + \rho_w g \nabla z\right) - \frac{\mathbf{k}_w^n(S_n)}{\mu_n}\cdot\left(\nabla p_n + \rho_n g \nabla z\right),$$

$$\mathbf{q}_{rn} = -\frac{\mathbf{k}_n^w(S_w)}{\mu_w}\cdot\left(\nabla p_w + \rho_w g \nabla z\right) - \frac{\mathbf{k}_n^n(S_n)}{\mu_n}\cdot\left(\nabla p_n + \rho_n g \nabla z\right),$$

Note: COUPLING, due to momentum transfer across the fluid-fluid interface.

For saturated flow: momentum transfer from fluid to solid: $-\frac{1}{\mathcal{U}_o}\int_{S_{ws}} \boldsymbol{\tau}\cdot\mathbf{n}\,dS$

- In the unsaturated zone, e.g., for the water

$$-\frac{1}{\mathcal{U}_o}\int_{S_{ws}} \boldsymbol{\tau}\cdot\mathbf{n}\,dS \quad \text{and} \quad -\frac{1}{\mathcal{U}_o}\int_{S_{wa}} \boldsymbol{\tau}\cdot\mathbf{n}\,dS$$

…and a similar sum for the air phase

- Many researchers claim that the coupling can be neglected, BUT not all of them, e.g., WALTER ROSE (see also the earlier works by Yuster (1951) and Odeh (1959) and many others in the Reservoir engineering field), but only a few experiments.

- Thus, we usually approximate:

$$\mathbf{q}_{rw} = -\frac{\mathbf{k}_w(S_w)}{\mu_w} \cdot \left(\nabla p_w + \rho_w g \nabla z\right),$$

$$\mathbf{q}_{rn} = -\frac{\mathbf{k}_n(S_n)}{\mu_n} \cdot \left(\nabla p_n + \rho_n g \nabla z\right),$$

…but still coupling between the two phases, due to

$$S_w + S_n = 1.0, \quad \text{and} \quad p_n - p_w = p_c(S_w)$$

In soil physics,

$$q_{rw} = -K_w(\psi) \cdot \nabla(\psi - z), \quad \psi = -\frac{p_w}{\rho_w g}$$

where ψ is called suction.

- Basic approach---each fluid occupies a certain portion of the void space, hence: DARCY"S LAW IS APPLICABLE TO EACH FLUID separately, with the effective permeability:

Common assumption

$$k_w = k_w(S_w), \quad k_n = k_n(S_n),$$

- Often it is ASSUMED, especially in unsaturated (air-water) flow , that the air is everywhere (more or less) at atmospheric pressure (not always justified!). Then, only:

$$\mathbf{q}_{rw} = -\frac{\mathbf{k}_w(S_w)}{\mu_w} \cdot \left(\nabla p_w + \rho_w g \nabla z\right), \quad p_a \approx 0, \quad \rho_a \approx 0.$$

Can be written in terms of water saturation, S_w

$$\mathbf{q}_{rw} = -\mathbf{D}_w(S_w) \cdot \nabla S_w - \mathbf{K}_w(S_w) \cdot \nabla z,$$

Moisture diffusivity $$\mathbf{D}_w = \mathbf{D}_w(S_w) = -\frac{\mathbf{K}_w(S_w)}{dS_w / d\psi}.$$

- The discussion of two-phase flow can be extended also to THREE FLUID PHASES,

- 3-phase flow used to be of interest primarily in reservoir engineering, but recently, because of subsurface contamination by NAPLs, this subject became important also for modeling subsurface (vadose zone and aquifer) contamination.

◆ ◆ ◆ ◆

..and I have not mentioned fractured rock domains…

- Although the focus of this presentation is on Darcy’s law, over the years, the field of TIPM has made huge progress also in many other areas. Obviously, Darcy’s law plays the essential role in all transport models.

- From groundwater flow → phenomena of transport in porous media

e.g.,

- DISPERSION solute transport, contaminant transport,…reactive transport, etc.
- Heat, ….., geothermal, ….energy storage,…..radioactive waste disposal, ,…
- Deformation….consolidation & land subsidence…wave propagation,…acoustics..
- Biomedical.. Transport of solutes in lungs and kidneys..

Obviously, during all this period, research and applications on:

- Groundwater hydrology
- Phenomena of transport in porous media

have been taking place and huge progress has been made., with applications in:

- Civil & Environmental Engineering and (groundwater flow, groundwater contamination and remediation, geo-mechanics, building materials)
- Agriculture Engineering (irrigation, drainage)
- Reservoir Engineering (improved recovery techniques, geothermal reservoirs)
- Biomedical engineering (modeling of fluids and solutes in body organs)
- Chemical Engineering (reactors and membranes)
- Radioactive waste disposal……etc.

- Significant developments in (e.g.,):

- Contaminant transport in the subsurface, with bio-chemical reactions. (with applications in remediation technique, radioactive waste repositories,…)

- Density dependent flow and transport (e.g., in connection with sea water intrusion.)

- Convective currents and instabilities for $\rho = \rho(p, c^\gamma, T)$

- Coupled transport phenomena (mass, heat, reactive solutes)

- Deformable porous media (and waves…acoustics, from TERZAGHI to BIOT….)

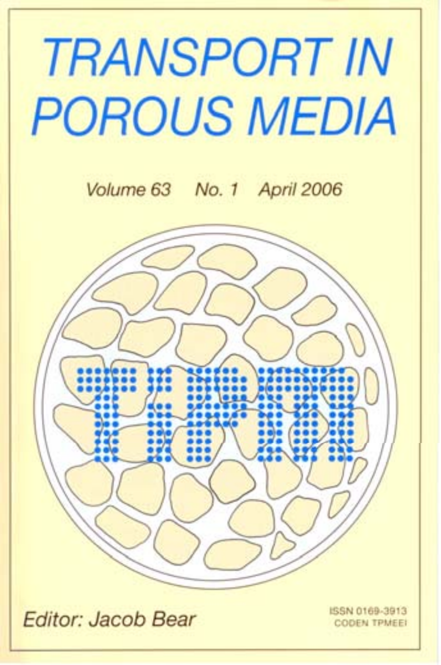
- Flow and transport in fractured (and porous) rocks,

- Biomedical engineering…

- Inverse problems (parameter identification)

- Upscaling, better understanding of multiple scale heterogeneity,

- Coping with uncertainty/heterogeneity… STOCHASTIC MODELING



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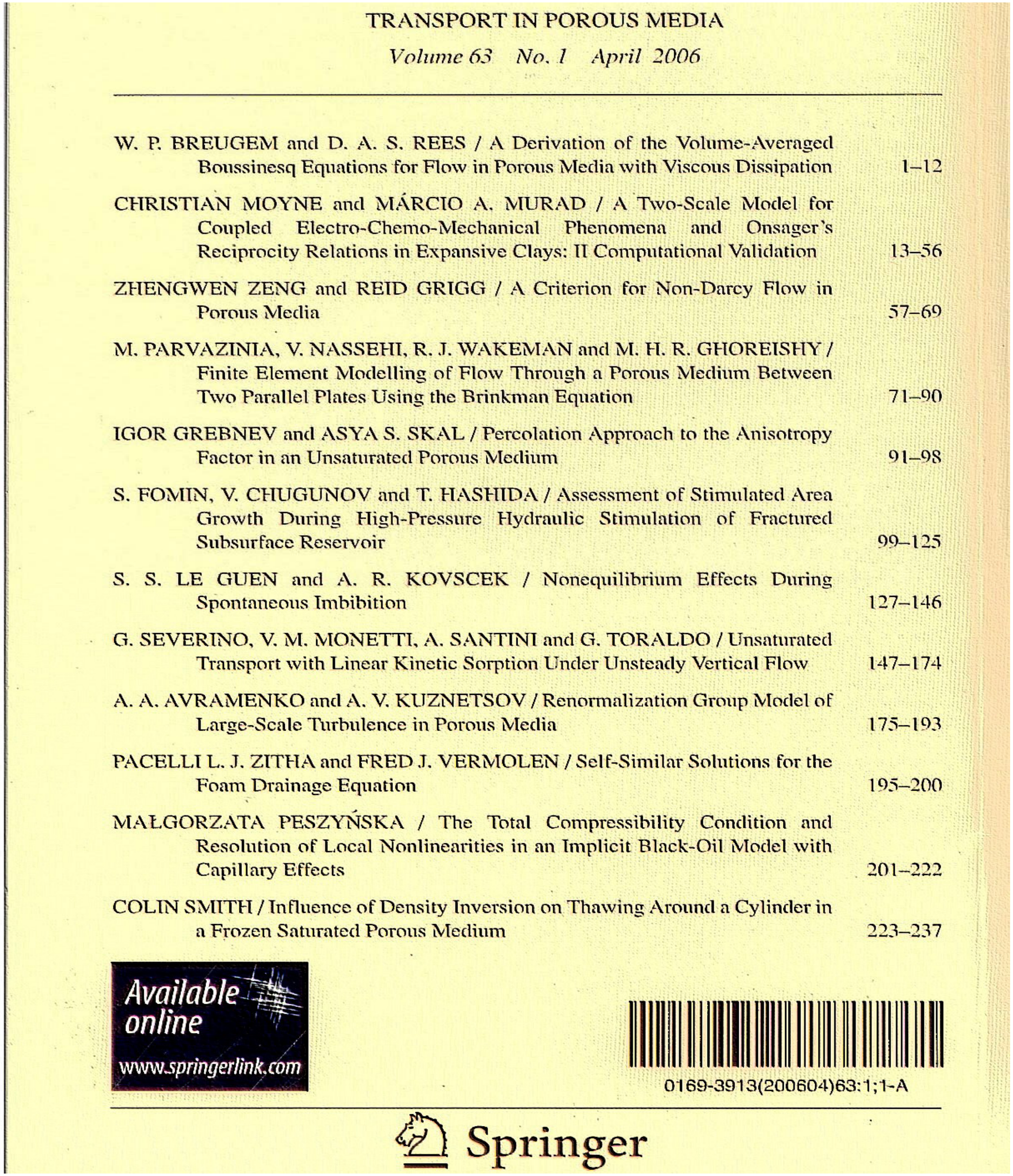

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Huge progress has been made in modeling:

- ❖Objective…..construct well-posed models of phenomena of transport,
- ❖Transform into NUMERICAL MODELS
- ❖Use computer codes (and powerful computers) for solving of models.

FOR RESEARCH AND PRACTICE

Quite some distance from ……………………………………

Darcy's (1856) experiments…and Dupuit assumption (1848, 1863)..
Boussinesq equation (1877, 1903-4, 1913)…. Forchheimer formula (1886, 1924-30)…
Badon-Ghyben law…(1888)… Buckingham (1907) ….. Gardner .(1920)..
Terzaghi (1925), Kozeny (1927)… Richards (1931)… Muskat (1932)..
Casagrande (1934-6).. .Theis (1935)… Childs (1936, 48))…
K. Hubbert (1940)…Day (1942)… Buckley & Leverett (1942)…Biot (1943, 1955-6)….
Edelfson (1943)… Polubarinova-Kochina (1942-51)
Brinkman (1947-9).. J.R.Philip (>1954)…G. De Josselin de Jong *(1955)..
Lykov (1954),..Carman (1950-1956) …Saffman (1957)…
Aravin & Numerov….

and many many others..

*May he live to 120!

So, where do we go from here?

(focusing mainly on groundwater flow and solute transport modeling).

❑The driving force will be increased demand for fresh, clean water (quantity and quality) for the population, and a clean environment.

❑Increased demand for more oil (enhanced oil production techniques.

❑Another driving force is the continuously growing power of computers which enable solutions of more complicated problems.

..and, of course…"publish or perish..

➢Modeling solute transport with multiple/many chemical and biological reactions, among multiple chemical species, not necessarily under equilibrium conditions, also in connection with natural attenuation and remediation techniques. From the microscopic level to the macroscopic one—to field scale. Modeling (chemical and biological) reactive transport, e.g., associated with artificial recharge of treated wastewater.

↑ Linked? ↓

➢Improved techniques for model calibration (inverse problem, parameter identification problem) for large scale fields (always heterogeneous) problems---both flow and solute transport coefficients. Uncertainty in calibration.

- Effects of pathways in both the saturated and the unsaturated zones, due to subsurface heterogeneity. Investigation techniques (e.g., tracers).

- Use of tracers and age determination for investigations of large water resources systems.

- More and improved methods for modeling under uncertainty in model parameters and coefficients, e.g., associated with the heterogeneity of the problem domain, or with replenishment from precipitation. Stochastic models. Probabilistic forecasting. Models for management under uncertainty.

- Efficient sat-unsat modeling for large scale aquifer domains, in view of the different space and time scales involved. Modeling of phenomena with different time scales.

- Improved efficient combined subsurface - above surface hydrological models for large watersheds. Improved estimates of natural replenishment.

- Simultaneous modeling of multiple scale flow and transport phenomena, e.g., domain with a porous solid matrix, and porous fracture rocks. The interaction (e.g., chemical/biological) between micropores and macropores (also microfractures/macrofractures)

- Combining models for the optimal management of groundwater resources with prediction models, which serve as a constraint to the optimization model.

- Efficient numerical techniques for cases in which phases and chemical species appear/disappear.

On the scale of water resources systems:

- Technologies to increase natural replenishment from precipitation.

- Technologies for remediation of contaminated subsurface.

- Optimal management (quality/quantity) of large scale water resource systems—with multiple surface water groundwater resources

- Effect of climate changes on water resources systems.

- Improved data management techniques, with linkage to large scale hydrogeological models.

On phenomena of transport in porous media.

- Chemical-biological reactions at micro level (including in micropores/microfractures) and their description at the macroscopic level.
- Heterogeneity at multiple scales, especially effect on solute and heat transport.
- Multiphase flow and transport (of heat and solute).
- Non-Newtonian fluids and non-elastic porous media.
- Transport of fines and colloids, with chemical/biological aspects. Movement of bacteria and viruses through the void space.
- Dispersion at different scales and in multiphase flow. Also in anisotropic porous media.
- Electric flow and electro-osmosis in porous media, in connection with remediation technologies.

- Use of heat in subsurface remediation.
- Compressed air energy storage, heat storage and sequestration of CO_2
- Dispersion in anisotropic porous media
- Dispersion in unsaturated/multiphase flow
- Anisotropy in unsaturated/multiphase flow

and…..hopefully…more fundamentals,….more experimental validation of models

….and a little less computer experiments…

THANK YOU